

Category Theory, Free Groups, Universal Properties, and Chatbots

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October 4, 2023

Abstract

While I was studying Category Theory, free groups in particular, I thought of a way to help me understand the notion of ‘universal property’ by defining what it means to be a group of order n by a commutative diagram and a universal property (definition). I wanted to get help from both BingChat and ChatGPT. What follows is my interaction with them.

1 Introduction

I don’t even know if what I came up with as a categorical description of a group of order n make sense within category theory, but one has to start somewhere. I looked to the two chatbots for help.

On 3 October 2023, I asked the following of both BingChat and ChatGPT:

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I need a categorical definition of a group of order n, provided
by a universal property from a bijective map on a set of order
n (f and f') each to two groups G, G', respectively, and a unique
homomorphism \phi from G to G' such that \phi f = f'.
Is this doable?
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Now, there’s more at stake here than whether or not their replies are technically correct or not. There’s the question of whether these LLMs understood the question posed to them and whether or not they replied with a reasonable response to it. Let’s see.

By the way, the first replies I show for both bots will be as close to their ASCII code math markup. Then, I’ll present their replies set in full LaTeX.

2 The Replies

First, ChatGPT replies:

Yes, it is indeed possible to define a group of order n using a universal property and a bijective map. The concept you are describing is closely related to a fundamental concept in group theory known as the “universal property of the free group.”

Let’s break down the definition step by step:

Start with a set of order n , which we’ll call X , having n elements.

Consider two groups, G and G' , and two bijective maps:

$$\begin{aligned} f: X &\rightarrow G \\ f': X &\rightarrow G \end{aligned}$$

The universal property you are describing involves the existence of a unique homomorphism φ from G to G' such that $\varphi f = f'$.

Here’s a categorical definition based on this idea:

Definition: A group of order n is a pair (G, f) , where G is a group, and f is a bijective map from a set X of order n to G , such that for any group G' and any bijective map f' from X to G' , there exists a unique group homomorphism $\varphi: G \rightarrow G'$ such that $\varphi f = f'$.

This definition essentially characterizes the group G up to isomorphism based on the bijective map f . In other words, any group G of order n , along with a bijective map f from X to G , uniquely determines G up to isomorphism, as it can be mapped via a homomorphism φ to any other group G' that also has a bijective map from X to G' .

This definition is in line with the notion of a group being determined by its universal property with respect to the given set X . It captures the essence of group structures based on the bijection between X and the group G .

This reply looks formatted in ASCII/Unicode. But when ChatGPT goes all out in formatting, it uses LaTeX/MathJax (I asked it). Anyway, now I’ll represent the above in full LaTeX:

Yes, it is indeed possible to define a group of order n using a universal property and a bijective map. The concept you are describing is closely related to a fundamental concept in group theory known as the “universal property of the free group.”

Let’s break down the definition step by step:

Start with a set of order n , which we’ll call X , having n elements.

Consider two groups, G and G' , and two bijective maps:

$$f : X \rightarrow G, \tag{1a}$$

$$f' : X \rightarrow G'. \tag{1b}$$

The universal property you are describing involves the existence of a unique homomorphism φ from G to G' such that $\varphi f = f'$.

Here's a categorical definition based on this idea:

Definition: A group of order n is a pair (G, f) , where G is a group, and f is a bijective map from a set X of order n to G , such that for any group G' and any bijective map f' from X to G' , there exists a unique group homomorphism $\varphi : G \rightarrow G'$ such that $\varphi f = f'$.

This definition essentially characterizes the group G up to isomorphism based on the bijective map f . In other words, any group G of order n , along with a bijective map f from X to G , uniquely determines G up to isomorphism, as it can be mapped via a homomorphism φ to any other group G' that also has a bijective map from X to G' .

This definition is in line with the notion of a group being determined by its universal property with respect to the given set X . It captures the essence of group structures based on the bijection between X and the group G .

Next, BingChat (same question):¹

Certainly! Let's delve into the categorical definition of a group of order n using the concept of a free group and its universal property.

1. Free Group (FS) Definition:

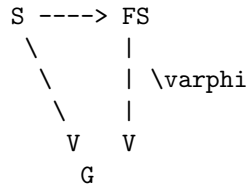
A free group FS over a set S is a pair (g, FS) that satisfies the universal property: For any group G and any map $f : S \rightarrow G$, there exists a unique group homomorphism $\varphi : FS \rightarrow G$ such that $\varphi \circ g = f$. In other words, FS is the "freest" group generated by the elements of S , allowing us to express any relation among these elements.

2. Universal Property Explained:

The universal property ensures that for any choice of group G and map $f : S \rightarrow G$, there exists a unique homomorphism $\varphi : FS \rightarrow G$ that makes the following diagram commute:

¹I asked BingChat what markup language it uses, and it said that it uses MathML, which needs HTML5.

[Note: These ‘V’s are meant to be arrowheads. I included both f and g on the diagram in Fig. 1.]



The map $g : S \rightarrow FS$ is required to be a one-to-one (injective) set-theoretic map. This ensures that distinct elements in S are mapped to distinct elements in FS . Essentially, we can think of g as an inclusion map. The uniqueness of φ guarantees that no matter which group G and map f we choose, there is only one way to extend the mapping from S to FS to a homomorphism from FS to G . This uniqueness is crucial for the universal property.

3. Why Is the Map g One-to-One?:

Suppose we have two distinct elements, say $s_1, s_2 \in S$, such that $g(s_1) = g(s_2)$. By the universal property, we obtain that $f(s_1) = f(s_2)$. However, since the map f is injective (one-to-one), this implies that $s_1 = s_2$. Therefore, we conclude that the map $g : S \rightarrow FS$ must also be one-to-one.

4. Isomorphism and Uniqueness:

Universal properties define objects up to isomorphism. While there may be many pairs (g, FS) satisfying the universal property, they are all isomorphic. By identifying the group and its image (which are isomorphic by the First Isomorphism Theorem), we ensure uniqueness up to isomorphism. In summary, the categorical definition of a group of order n , based on a universal property from a bijective map on a set of order n , leads us to the concept of free groups. These free groups capture essential algebraic properties without imposing additional relations beyond those specified by the set itself¹².

References provided by BingChat:

1. <https://math.stackexchange.com/questions/130850/free-group-and-universal-property>
2. <https://sites.math.northwestern.edu/~scanez/courses/395/notes/lecture-notes-category-theory.pdf>

Next, BingChat’s reply in full LaTeX:

Certainly! Let’s delve into the categorical definition of a group of order n using the concept of a free group and its universal property.

1. Free Group (FS) Definition:

A free group FS over a set S is a pair (g, FS) that satisfies the universal property: For any group G and any map $f : S \rightarrow G$, there exists a unique group homomorphism $\varphi : FS \rightarrow G$ such that $\varphi \circ g = f$. In other words, FS is the “freest” group generated by the elements of S , allowing us to express any relation among these elements.

2. Universal Property Explained:

The universal property ensures that for any choice of group G and map $f : S \rightarrow G$,² there exists a unique homomorphism $\varphi : FS \rightarrow G$ that makes the following diagram commute:

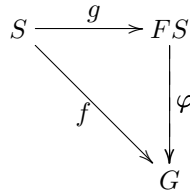


Figure 1. The diagram done in LaTeX.

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²I think that BingChat should have declared f to be any one-to-one map.

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2. <https://sites.math.northwestern.edu/~scanez/courses/395/notes/lecture-notes-category-theory.pdf>

3 Conclusion

Okay, I don't know if either chatbot gave me a correct reply, but both replies were responsive to my question. I have to give them that much. ChatGPT worked most closely with my own suggestion, and this seems to be a pattern with it. BingChat seems to go much further afield in replying to me. In this case, it seems that BingChat wanted to produce the result I was going for as a mere corollary to the universal description of a free group, but I don't know if that's the correct way to go.