

Function Composition is Associative

P. Reany

April 7, 2023

Abstract

What follows is a proof that the composition of functions is associative as done by ChatGPT. I converted the plain text output I got from ChatGPT into LaTeX.

User: I need a proof that the composition of functions is associative, please.¹

ChatGPT: Sure, I can provide a proof that the composition of functions is associative.

Let's consider three functions: f , g , and h , and let's assume that they are all composable. That means, the output of $f(x)$ can be used as the input of $g(x)$, and the output of $g(x)$ can be used as the input of $h(x)$.

Now, let's take the composition of these functions in two different ways:

1. $f(g(h(x)))$ – This means we first apply $h(x)$, then take the output and apply $g(x)$, and finally take the output of $g(x)$ and apply $f(x)$.
2. $(f(g))(h(x))$ – This means we first apply $g(x)$ and $f(x)$ together (the composition of $f(x)$ and $g(x)$), and then take the output of that and apply $h(x)$.

To show that the composition of functions is associative, we need to prove that these two different ways of composing the functions give the same result.

Let's start with the first way of composing the functions, $f(g(h(x)))$:

$$f(g(h(x))) = f(g(h(x))) = f(g(h(x))) \tag{1}$$

Now, let's consider the second way of composing the functions, $(f(g))(h(x))$:

$$(f(g))(h(x)) = f(g(h(x))) = f(g(h(x))) \tag{2}$$

As we can see, both ways of composing the functions give us the same result, which means that the composition of functions is associative. We can generalize this result for any number of composable functions.

Equations (1) and (2) look weirdly redundant, but they will make more sense when compared to Equations (4) and (5) below.

¹ChatGPT proves that function composition is associative (4/6/2023).

User: How did you arrive at this proof?

ChatGPT: This proof is a standard one used to show the associativity of composition of functions.

To prove associativity, we need to show that the order of applying functions does not matter. That is, if we have three functions f , g , and h , then applying f after g after h , or applying f after $(g$ after $h)$, gives us the same result.

To show this, we need to start by defining what we mean by the composition of functions. The composition of two functions f and g is denoted by $(f \circ g)$, and is defined as follows:

$$(f \circ g)(x) = f(g(x)) \tag{3}$$

In other words, we take the output of $g(x)$ and use it as the input for $f(x)$. We can extend this definition to any number of functions.

Now, to prove associativity, we consider three functions f , g , and h . We want to show that $(f \circ g) \circ h = f \circ (g \circ h)$.

We start by evaluating $(f \circ g) \circ h$ for an arbitrary input x :

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))) \tag{4}$$

Next, we evaluate $f \circ (g \circ h)$ for the same input x :

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x))) \tag{5}$$

As we can see, the results are the same. Therefore, we have shown that the order of applying functions does not matter, and the composition of functions is associative.

It's interesting that the second time ChatGPT presented this proof, it used the composition symbol \circ , which makes the proof a bit easier to follow.

1 ChatGPT with LaTeX

I asked ChatGPT to present the same proof but in LaTeX code. This is what it gave back:

Let x be an arbitrary element in the domain of f , g , and h . Then we have:

```
\begin{align*}
((f \circ g) \circ h)(x) &= (f \circ g)(h(x)) \tag{definition of composition} \\
&= f(g(h(x))) \tag{definition of composition} \\
&= (f \circ (g \circ h))(x) \tag{definition of composition}
\end{align*}
```

This code isn't correct LaTeX. After I made some adjustments, I ran it to get:

Let x be an arbitrary element in the domain of f , g , and h . Then we have:

$$\begin{aligned} ((f \circ g) \circ h)(x) &= (f \circ g)(h(x)) \\ &= f(g(h(x))) \\ &= (f \circ (g \circ h))(x) \end{aligned}$$

The logic of this terse proof is (almost) correct.² I explain why the proof is so short in the Conclusion.

2 Conclusion

The triviality of this proof rest on the two defining aspects of function composition, in which 1)

$$(f \circ g)(x) \equiv f(g(x)), \tag{6}$$

and 2)

$$(f \circ (g \circ h))(x) \equiv f((g \circ h)(x)), \tag{7}$$

or, the outermost composition being evaluated first, and then you apply the first rule.

By the way, I gave BingChat a chance to prove this theorem, but it did not do nearly so well as ChatGPT. Maybe soon it will do better.

²More correct is to say that x is in the domain of h .