2×2 Matrices and 1st-order Linear DEQs with Isotropic Spinors

P. Reany

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Abstract

In this paper I show some uses of isotropic spinors in the search for eigenvalues and eigenvectors of 2×2 matrices. The most significant results are first a new approach to Cayley-Hamilton Theorem and also a new approach to the theory of coupled first-order differential equations. Also included is a proof of Liouville's Formula.

1 Introduction:

Here we investigate a few fundamentals of matrices, such as matrix inverses and eigenvalue/vectors. Our treatment will be restricted to 2×2 matrices, but the proofs have a beauty that's worth the small effort needed to appreciate them.

Now, you're probably familiar with the Euclidean inner product of two vectors, but we aren't going to use that inner product much. But for comparison's sake, we'll show what the Euclidean inner product of two vectors B^α, A^α looks like:

$$B_\alpha A^\alpha \equiv \sum_\alpha B_\alpha A^\alpha = B_1 A^1 + B_2 A^2 \,. \tag{1}$$

In other words, to scalar-multiply two column vectors together, we will have to convert one of them to a row vector by this procedure:

$$B_\alpha \equiv [J B^\alpha]^t = (B^\alpha)^t J^t \,, \tag{2}$$

where J is a 2×2 matrix used to lower the index of a vector. For the Euclidean case, J is the 2×2 identity matrix, and

$$B_\alpha \mapsto (B_1, B_2) \equiv (B^1, B^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (B^1, B^2) \,, \tag{3}$$

with the resulting familiar Euclidean square

$$B_\alpha B^\alpha = (B^1, B^2) \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} = (B^1)^2 + (B^2)^2 \,. \tag{4}$$

However, for our symplectic inner product, we take the symplectic 2×2 matrix

$$J \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5}$$

and get

$$B_\alpha \mapsto (B_1, B_2) \equiv (B^1, B^2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (B^2, -B^1)\,. \tag{6}$$

And if we multiply the above equation through by J and reverse sides, we have that

$$(B^1, B^2) = (-B_2, B_1)\,. \tag{7}$$

Now we're ready to take the symplectic 'inner' product of vectors B^α and A^α:

$$B_\alpha A^\alpha = (B_1, B_2) \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} = (B^2, -B^1) \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} = B^2 A^1 - B^1 A^2 \tag{8}$$

This inner product acts like a determinant:

$$B_\alpha A^\alpha = \begin{vmatrix} A^1 & A^2 \\ B^1 & B^2 \end{vmatrix} = B^2 A^1 - B^1 A^2 \tag{9}$$

And, on taking this inner product in the opposite order, we get

$$A_\alpha B^\alpha = A^2 B^1 - A^1 B^2 = -(B^2 A^1 - B^1 A^2) \tag{10}$$

Comparing (9) to (10), we see that they are negatives of each other

$$A_\alpha B^\alpha = -B_\alpha A^\alpha\,. \tag{11}$$

So our symplectic inner product acts like the antisymmetric cross product of two vectors.

Raising and lowering indicies To see more on raising and lowering indicies of spinors, see Appendices 1 and 2.

Definition: A nonzero vector whose 'square' is zero is said to be **isotropic**.

Definition: So far the components of our two-component vectors have been real numbers or real-valued functions. But now we let them be complex-valued, in which case we refer to them as **spinors**.

Definition: If the product of two nonzero vectors/spinors is zero, we shall refer to the product as an **isotrope**.

Definition:
A **paraquant** is an equation constructed similar to a given equation to solve, which if it can be solved, provides information useful to solve the given equation. Example: Given the original equation $L(y) = f(x)$, where L is a linear operator, the **method of Green functions** attempts to solve the paraquant equation $L[G(x,\xi)] = \delta(x - \xi)$, to get $y = \int G(x,\xi) f(\xi) d\xi$.

2 The Main Heuristic

Let X^α and Y^α be nonzero spinors forming the isotrope $Y_\alpha X^\alpha = 0$; then we know that there exists a nonzero scale factor κ, say, such that

$$Y_\alpha = \kappa X_\alpha \,. \tag{12}$$

Say we wish to solve for κ. One way to do this is to have a third spinor Z^α satisfiying two equations

$$Y_\alpha Z^\alpha = a \,, \tag{13a}$$

$$X_\alpha Z^\alpha = b \,. \tag{13b}$$

Then we multiply (12) through by Z^α and sum, yielding the relation $a = \kappa b$ to solve for κ.

The Determinant of A:
Let A be the matrix

$$A = \begin{bmatrix} r_1 & r_2 \\ R_1 & R_2 \end{bmatrix} = \begin{bmatrix} -r_\alpha- \\ -\, R_\alpha- \end{bmatrix} , \tag{14}$$

where $r_\alpha \mapsto (r_1, r_2)$ and $R_\alpha \mapsto (R_1, R_2)$ as

$$\begin{aligned} r^\alpha R_\alpha &= r^1 R_1 + r^2 R_2 = -r_2 R_1 + r_1 R_2 \\ &= \det \begin{bmatrix} r_1 & r_2 \\ R_1 & R_2 \end{bmatrix} = \det(A) = |A| \,. \end{aligned} \tag{15}$$

But, using (11), we get that

$$r_\alpha R^\alpha = -\det(A) = -|A| \,. \tag{16}$$

Lemma:
Now, let $A\mathbf{x} = \mathbf{0}$ but $\mathbf{x} \neq \mathbf{0}$, then $|A| = 0$.

Proof:

$$A\mathbf{x} = \begin{bmatrix} r_\alpha \\ R_\alpha \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} r_\alpha \\ R_\alpha \end{bmatrix} x^\alpha = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \tag{17}$$

From which we get that

$$r_\alpha x^\alpha = 0, \qquad R_\alpha x^\alpha = 0 \,. \tag{18}$$

But we can claim that (from the Main Heuristic)

$$x_\alpha = c_1 r_\alpha, \qquad x^\alpha = c_2 R^\alpha \,, \tag{19}$$

where neither c_1 nor c_2 is zero. But from the identity $x_\alpha x^\alpha = 0$, we get that $c_1 c_2 r_\alpha R^\alpha = 0$, implying that

$$r_\alpha R^\alpha = -|A| = 0 \,. \tag{20}$$

Hence, $|A| = 0$.

Inverse of a 2×2 Matrix A.
Given an arbitrary 2×2 matrix A, find a 2×2 matrix B that acts as a right inverse to A. That is,

$$AB = \mathrm{I}\,, \tag{21}$$

where I is the 2×2 identity matrix.

Solution: Since we'll be doing row by column multiplication, let's set-up A as two rows of spinors and B as two columns of spinors:

$$AB = \begin{bmatrix} -r_\alpha- \\ -\,R_\alpha- \end{bmatrix} [w^\alpha, W^\alpha] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\,. \tag{22}$$

Multiplying these out, we get the two isotrope equations for the off-diagonal terms:

$$R_\alpha w^\alpha = 0, \qquad r_\alpha W^\alpha = 0\,, \tag{23}$$

which we can use to solve for w^α, W^α up to scale factors

$$w^\alpha = \kappa R^\alpha, \qquad W^\alpha = \kappa' r^\alpha\,, \tag{24}$$

So, this is the beauty of isotropic spinors: We've only just started this solution, yet we're nearly finished. All we need to do now is to solve for the κ's by plugging the values of w^α, W^α from (24) into the diagonal terms from (22):

$$r_\alpha w^\alpha = 1, \qquad R_\alpha W^\alpha = 1\,. \tag{25}$$

The results are: $\kappa = -\dfrac{1}{|A|}$ and $\kappa' = \dfrac{1}{|A|}$. Therefore,

$$B = -\frac{1}{|A|}[R^\alpha, -r^\alpha] = -\frac{1}{|A|}\begin{bmatrix} R^1 & -r^1 \\ R^2 & -r^2 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} R_2 & -r_2 \\ -\,R_1 & r_1 \end{bmatrix}\,, \tag{26}$$

which is the conventional form of the solution. For the completion of this proof (to find the left inverse to A), see **Appendix 3**.

See **Appendix 4** for the solution to solve for x^α in the equation $Ax^\alpha = y^\alpha$.

One Easy Lemma:
It's useful at times to express the columns of the 2×2 matrix A in terms of its symplectic conjugate rows. To that end, let t^α be the first column of A, and let T^α be the second column. Then, I state without proof that

$$t^\alpha = \begin{bmatrix} T_A \\ 0 \end{bmatrix} + R^\alpha \quad \text{and} \quad T^\alpha = \begin{bmatrix} 0 \\ T_A \end{bmatrix} - r^\alpha\,, \tag{27}$$

where T_A is the trace of A. Thus,

$$A = [-R^\alpha, r^\alpha] + T_A\mathrm{I}\,. \tag{28}$$

3 The Cayley-Hamilton Theorem:

In standard linear algebra, the process of finding eigenvalues and eigenvectors of a transformation begins with finding the determinant of the matrix $\lambda\mathrm{I} - A$, resulting in the so-called *characteristic polynomial*

$$p(\lambda) = \lambda^2 - T_A\lambda + |A|\,. \tag{29}$$

Then we set this polynomial equal to zero to solve for the roots to λ, the so-called eigenvalues of the *characteristic equation*. The Cayley-Hamilton Theorem states that the polynomial equation $p(A) = \mathbf{0}$, where $\mathbf{0}$ is the 2×2 matrix of zeros for entries. In other words,

$$A^2 - T_A A + |A|\mathrm{I} = \mathbf{0}\,. \tag{30}$$

Proof:
Multiply Eq. (28) through by $A = \begin{bmatrix} r_\alpha \\ R_\alpha \end{bmatrix}$ and simplify!

Corollary:
Using Cayley-Hamilton and the fact that the determinant of a product of 2×2 matrices is the product of their determinants (proved in Appendix 8), it is easy to prove that the trace of a 2×2 matrix A is invariant under a similarity transformation: $A \mapsto M^{-1}AM$.

Proof:
Let $B = M^{-1}AM$. Then, multiplying (30) on the left by M^{-1} and on the right by M, and making the standard virtual emplacements, we arrive at

$$B^2 - T_A B + |A|\mathrm{I} = \mathbf{0}\,. \tag{31}$$

On replacing $|A|$ by $|B|$, we get

$$B^2 - T_A B + |B|\mathrm{I} = \mathbf{0}\,. \tag{32}$$

From this we conclude that $T_A = T_B$.

Definition 1:
We define the *Wronskian* function $W(t)$ by

$$W(t) = \phi_\alpha \dot{\phi}^\alpha\,. \tag{33}$$

Theorem 1:
Let the transformation be

$$\dot{\phi}^\alpha = A\phi^\alpha\,, \tag{34}$$

where the entries of A are not functions of time.

Then,

$$W(t) = W_0 e^{T_A t}\,, \tag{35}$$

where $W = W_0$ at $t = 0$.
Proof:
Differentiating (34), we get

$$\ddot{\phi}^\alpha = A\dot{\phi}^\alpha = A^2\phi^\alpha\,. \tag{36}$$

Differentiating (33), we get

$$\begin{aligned}
\dot{W}(t) \quad &= \phi_\alpha \ddot{\phi}^\alpha && \text{(since } \dot{\phi}_\alpha \dot{\phi}^\alpha = 0\text{)} \\
&= \phi_\alpha A^2 \phi^\alpha && \text{(from (36))} \\
&= \phi_\alpha [T_A A - |A|\mathbf{I}]\phi^\alpha && \text{(from Cayley-Hamilton)} \\
&= T_A \phi_\alpha A \phi^\alpha - |A|\phi_\alpha\phi^\alpha && \text{(this 2nd term is zero)} \\
&= T_A \phi_\alpha \dot{\phi}^\alpha && \text{(from (34))} \\
&= T_A W && \text{(from (33))}\,.
\end{aligned}$$

Rearranging, we get

$$\frac{dW}{W} = T_A dt\,. \tag{37}$$

Integrating this from time 0 to t, we get (35). And we are done.

Of course, for traceless matrices A, we get that W is a constant, which should be of some use when working in the Pauli Algebra.

Theorem 2:
Returning to the Wronskian $W(t)$, it's a simple matter to show that

$$\frac{W(t)}{\phi^1\phi^2} = \frac{d}{dt}\ln\left[\frac{\phi^2}{\phi^1}\right]\,. \tag{38}$$

Proof:
Left to the reader.

Equation (35) constitutes a relation on the components of a given spinor solution to (34), and, as we shall soon see, if A is given then we can easily find both linearly independent spinor solutions to (34) without use of (35). However, maybe if instead of A being given, only its trace is given, then (35) may be of use.

Finally, I'll present one more pretty computation generating a result of dubious value. Suppose that A has the form

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\,. \tag{39}$$

Then

$$\begin{aligned} W(t) &= \phi_\alpha \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \phi^\alpha \\ &= \phi_\alpha \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \phi^\alpha + \phi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \phi^\alpha \\ &= \lambda \cancelto{0}{\phi_\alpha \phi^\alpha} + [\phi^2, -\phi^1] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \\ &= (\phi^2)^2 \,. \end{aligned}$$

Therefore,

$$W_0 e^{2\lambda t} = -(\phi^2)^2 \,. \tag{40}$$

Theorem 3:
The following theorem, known as **Liouville's Formula**, is presented in two dimensions, but is doable in n dimensions. However, I will not be using the spinor method for the proof and I'll explain why when the proof is over.

Let the transformation be

$$\dot{\phi}^\alpha = A(t)\phi^\alpha \,, \tag{41}$$

where the entries of A are functions of time. We will suppose that this equation has two linearly independent solutions ϕ_1^α, ϕ_2^α. Let Φ be a function of time, defined as

$$\Phi(t) \equiv [\phi_1^\alpha, \phi_2^\alpha] = \begin{bmatrix} \phi_1^1 & \phi_2^1 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} \,. \tag{42}$$

Define

$$\Gamma(t) \equiv \det(\Phi(t)) \,. \tag{43}$$

Then

$$\Gamma(t) = \Gamma_0 e^{\int T_A dt} \,, \tag{44}$$

where T_A is the trace of A and $\Gamma(0) = 0$.

Proof:
We begin by writing (41) in the form of

$$\dot{\phi}^\alpha = \begin{bmatrix} r_1 & r_2 \\ R_1 & R_2 \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \,. \tag{45}$$

Therefore

$$\dot{\phi}^1 = r_\alpha \phi^\alpha \quad \text{and} \quad \dot{\phi}^2 = R_\alpha \phi^\alpha \quad \text{and} \quad T_A = r_1 + R_2 \,. \tag{46}$$

Differentiating (43), we get

$$\begin{aligned} \dot{\Gamma} &= \det \begin{bmatrix} \dot{\phi}_1^1 & \dot{\phi}_2^1 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} + \det \begin{bmatrix} \phi_1^1 & \phi_2^1 \\ \dot{\phi}_1^2 & \dot{\phi}_2^2 \end{bmatrix} \\ &= D_1 + D_2 \,. \end{aligned} \tag{47}$$

Note:

$$D_1 = \det \begin{bmatrix} \dot{\phi}_1^1 & \dot{\phi}_2^1 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} = \det \begin{bmatrix} r_1\phi_1^1 + r_2\phi_1^2 & r_1\phi_2^1 + r_2\phi_2^2 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} \tag{48}$$

We need a bit of matrix theory to understand the next key step. Adding or subtracting a nonzero multiple of one line into another line of a determinant will not change the value of the determinant. So, let's subtract r_2 times line 2 from line 1 of the above determinant, to get

$$D_1 = \det \begin{bmatrix} r_1\phi_1^1 & r_1\phi_2^1 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} = r_1 \det \begin{bmatrix} \phi_1^1 & \phi_2^1 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} = r_1\Gamma\,. \tag{49}$$

Doing a similar calculation for D_2, we get

$$D_2 = R_2\Gamma\,. \tag{50}$$

From these last two results, (47) becomes

$$\dot{\Gamma} = D_1 + D_2 = r_1\Gamma + R_2\Gamma = T_A\Gamma\,. \tag{51}$$

Rearranging, we get

$$\frac{d\Gamma}{\Gamma} = T_A dt\,. \tag{52}$$

Integrating this from time 0 to t, we get (44). And we are done.

So why didn't I do the proof in the spinors formalism? Actually, I did for myself to prove that it works. But it is less illuminating than linear algebra determinant formalism. The reason is that the derivatives of the determinant were taken across the rows and not of the columns, which are the solution spinors to the differential equation. So, I treated the rows of Φ as spinors and proceeded. It worked, but the proof was not any shorter than what I presented above. Maybe I just missed a better proof.

By the way, proving this theorem that the value of a determinant is unchanged by adding a multiple of one row to another row is simple using isotropic spinors. For example, consider the determinant of the two-column spinors A^α and B^α, which is expressible as $D = A_\alpha B^\alpha$. This looks like

$$\det \begin{bmatrix} A^1 & A^2 \\ B^1 & B^2 \end{bmatrix}. \tag{53}$$

Adding a multiple of B^α, say τB^α, to A^α within the determinant would simply be

$$-A_\alpha B^\alpha \mapsto -(A_\alpha + \tau B_\alpha)B^\alpha = -A_\alpha B^\alpha - \tau B_\alpha B^\alpha = -A_\alpha B^\alpha\,. \tag{54}$$

And that finishes the proof.

4 The Method of Undetermined Coefficients — Easy Version:

Let the transformation be

$$\dot{\phi}^\alpha = A\phi^\alpha \,, \tag{55}$$

where A is upper triangular with constant entries and $r_1 \neq R_2$:

$$A = \begin{bmatrix} -r_\alpha- \\ -R_\alpha- \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ 0 & R_2 \end{bmatrix} . \tag{56}$$

A reasonable ansatz for ϕ^α is

$$\phi^\alpha = M r^\alpha + N R^\alpha \,, \tag{57}$$

where M and N are scalar functions of time to be solved for. Plugging (57) into (55) yields

$$\dot{M} r^\alpha + \dot{N} R^\alpha = |A| \begin{bmatrix} -N \\ M \end{bmatrix} . \tag{58}$$

Multiplying this through first by r_α yields

$$\dot{N} = r_1 N - r_2 M \,, \tag{59}$$

and then multiplying (58) by R_α, gives

$$\dot{M} = R_2 M \,, \tag{60}$$

since $R_1 = 0$, whose solution is

$$M(t) = M_0 e^{R_2 t} \,. \tag{61}$$

Plugging (61) into (59) yields

$$\dot{N} - r_1 N = -r_2 M_0 e^{R_2 t} \,, \tag{62}$$

whose general solution is

$$N(t) = c e^{r_1 t} + \frac{r_2}{r_1 - R_2} M_0 e^{R_2 t} \,, \tag{63}$$

where c is a constant of integration. Hence, the general solution to (55) is

$$\phi^\alpha(t) = M_0 e^{R_2 t} r^\alpha + \big[c e^{r_1 t} + \frac{r_2}{r_1 - R_2} M_0 e^{R_2 t}\big] R^\alpha \,, \tag{64}$$

This gets expanded to

$$\begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} = M_0 e^{R_2 t} \begin{bmatrix} r_2 \\ -r_1 \end{bmatrix} + \big[c e^{r_1 t} + \frac{r_2}{r_1 - R_2} M_0 e^{R_2 t}\big] \begin{bmatrix} R_2 \\ 0 \end{bmatrix} \,, \tag{65}$$

I tested this solution for (55/56) on the equation

$$\dot{\phi}^\alpha = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} \phi^\alpha \,, \tag{66}$$

and it worked.

5 The Method of Undetermined Coefficients — Full Version:

Let the transformation be

$$\dot{\phi}^\alpha = A\phi^\alpha \,, \tag{67}$$

where A is a general matrix with differentiable time-dependent entries:

$$A = \begin{bmatrix} -r_\alpha- \\ -R_\alpha- \end{bmatrix} = \begin{bmatrix} r_1(t) & r_2(t) \\ R_1(t) & R_2(t) \end{bmatrix} . \tag{68}$$

Again, a reasonable ansatz for ϕ^α is

$$\phi^\alpha = Mr^\alpha + NR^\alpha \,, \tag{69}$$

where M and N are scalar functions of time to be solved for. Plugging (69) into (67) yields

$$\begin{aligned} \dot{M}r^\alpha + M\dot{r}^\alpha + \dot{N}R^\alpha + N\dot{R}^\alpha &= \begin{bmatrix} r_\alpha \\ R_\alpha \end{bmatrix} [Mr^\alpha + NR^\alpha] \\ &= |A| \begin{bmatrix} -N \\ M \end{bmatrix} . \end{aligned} \tag{70}$$

Multiplying (70) through first by r_α yields

$$Mr_\alpha\dot{r}^\alpha + \dot{N}r_\alpha R^\alpha + Nr_\alpha\dot{R}^\alpha = |A|[\,-r_1N + r_2M] \,, \tag{71a}$$

and then by R_α, gives

$$\dot{M}R_\alpha r^\alpha + MR_\alpha\dot{r}^\alpha + NR_\alpha\dot{R}^\alpha = |A|[\,-R_1N + R_2M] \,. \tag{71b}$$

Using that $r_\alpha R^\alpha = |A|$, these last two equations become

$$Mr_\alpha\dot{r}^\alpha + |A|\dot{N} + Nr_\alpha\dot{R}^\alpha = |A|[\,-r_1N + r_2M] \,, \tag{72a}$$

and

$$-|A|\dot{M} + MR_\alpha\dot{r}^\alpha + NR_\alpha\dot{R}^\alpha = |A|[\,-R_1N + R_2M] \,. \tag{72b}$$

Now we must simplify by setting a constraint (similar to that which is done in the variation of parameters problem): From (72a) we set[1]

$$\dot{N} = -r_1N + r_2M \,, \tag{73a}$$

leaving

$$Mr_\alpha\dot{r}^\alpha + Nr_\alpha\dot{R}^\alpha = 0 \,. \tag{73b}$$

[1] Why would we do this? Well, we must reduce the complexity of the system of equations to solve simultaneously, and this particular choice of constraint not only produces a 'simple' pair of equations to deal with, it also eliminates having to deal with $|A|$ in the resulting system.

And from (72b) we set

$$\dot{M} = R_1 N - R_2 M \,, \tag{74a}$$

leaving

$$M R_\alpha \dot{r}^\alpha + N R_\alpha \dot{R}^\alpha = 0 \,. \tag{74b}$$

On eliminating M between (73a) and (73b), we get

$$\dot{N} = p(t) N \,, \tag{75a}$$

where

$$p(t) = -\Big(r_1 + r_2 \frac{r_\alpha \dot{R}^\alpha}{r_\beta \dot{r}^\beta} \Big) \,, \tag{75b}$$

and $r_\beta \dot{r}^\beta \neq 0$ added to the constraints on the matrix A. This last equation has solution

$$N(t) = c_2 e^{\int p(t)\, dt} \,, \tag{75c}$$

where c_2 is a constant of integration. Similarly, on eliminating N between (74a) and (74b), we get

$$\dot{M} = q(t) M \,, \tag{76a}$$

where

$$q(t) = -\Big(R_2 + R_1 \frac{R_\alpha \dot{r}^\alpha}{R_\beta \dot{R}^\beta} \Big) \,, \tag{76b}$$

and $R_\beta \dot{R}^\beta \neq 0$ added to the constraints on the matrix A. This last equation has solution

$$M(t) = c_1 e^{\int q(t)\, dt} \,. \tag{76c}$$

where c_1 is also a constant of integration. Therefore, the solution for (69) is

$$\phi^\alpha(t) = c_1 e^{\int q(t)\, dt} r^\alpha + c_2 e^{\int p(t)\, dt} R^\alpha \,. \tag{77}$$

Warning: Do not confuse the explicit alphas in this last equation with the implicit alphas being summed on within $q(t)$ and $p(t)$.

We have one final consistency check to perform before leaving this problem. Equations (73b) and (74b) are two equations in M and N, and as such we must find out what restrictions they place on r_α and R_α (and hence on the rows of the matrix A). On eliminating M and N between (73b) and (74b), we end up with (remembering that we're summing on α and β)

$$r_\alpha R_\beta [\dot{r}^\alpha \dot{R}^\beta - \dot{R}^\alpha \dot{r}^\beta] = 0 \,. \tag{78}$$

But this equation is true for all r_α and R_α because $r_\alpha R_\beta$ is symmetric in α and β, while the expression inside the brackets is antisymmetric in them. Thus, the equation is true for all r_α and R_α.

6 The Method of Undetermined Coefficients — Special Case:

Let the transformation be as in (67), but with $R_1 = 0$, that is, A is upper triangular. We also require that $|A|$ be nonsingular. Hence, we immediately conclude two facts:

1. $\dot{R}_1 = 0$ and $R_\alpha \dot{R}^\alpha = 0$,
2. $|A| = r_1 R_2 \neq 0$.

Using these values in (72b), we get

$$-|A|\dot{M} + MR_\alpha \dot{r}^\alpha = -|A|R_2 M\,, \tag{79}$$

the importance of which is that all terms involving N have dropped out. Dividing this through by $-|A|$ and simplifying, we have

$$\dot{M} + q(t)M = 0\,. \tag{80a}$$

where

$$q(t) = -(\frac{\dot{r}_1}{r_1} - R_2) = -(D_t \ln r_1 - R_2)\,. \tag{80b}$$

So, (80a) has solution

$$M(t) = M_0 e^{-\int q(t)dt} = \frac{M_0}{r_1} e^{\int R_2 dt}\,. \tag{81}$$

Then, using this result in (73b), we have,

$$\dot{N}(t) + p(t)N = f(t)\,, \tag{82a}$$

where

$$p(t) = \frac{\dot{R}_2}{R_2} - r_1 = D_t \ln R_2 - r_1\,, \tag{82b}$$

and

$$f(t) = [\frac{r_2\dot{r}_1 - r_1\dot{r}_2}{r_1 R_2} - r_2]M(t)\,. \tag{82c}$$

Similar to (81), we note that

$$e^{\int p(t)dt} = R_2 e^{-\int r_1 dt}\,. \tag{83}$$

Of course, by formula, the solution to (82a) is

$$N(t) = e^{-\int p(t)dt}[\int^t e^{\int p(\tau)d\tau} f(\tau)d\tau + C]\,, \tag{84}$$

where C is a constant. So, we plug these results into (69) to get

$$\begin{bmatrix}\phi^1(t)\\ \phi^2(t)\end{bmatrix} = M(t)\begin{bmatrix} r_2 \\ -r_1\end{bmatrix} + N(t)\begin{bmatrix}R_2\\ 0\end{bmatrix}\,. \tag{85}$$

7 The Method of Undetermined Coefficients — Special Case Generalized:

If we take a general matrix A in (67) and (68) and convert it to an upper-triangular matrix, we can then use the result of the last section. To that end, we start with (67), where A is a general 2×2 matrix with time-dependent entries, and convert it to

$$\dot{\psi}^\alpha = U\psi^\alpha \,, \tag{86}$$

where

$$U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} \,, \tag{87a}$$

and

$$\psi^\alpha = G\phi^\alpha \,, \tag{87b}$$

where G is an invertible matrix to be determined, and, of course,

$$\dot{\psi}^\alpha = \dot{G}\phi^\alpha + G\dot{\phi}^\alpha \,. \tag{88}$$

So, we start by multiplying (67) through by G and inserting a virtual emplacement of $G^{-1}G$,[2]

$$G\dot{\phi}^\alpha = GAG^{-1}G\phi^\alpha \,. \tag{89}$$

Adding $\dot{G}\phi^\alpha$ to both sides, we get

$$\dot{G}\phi^\alpha + G\dot{\phi}^\alpha = \dot{G}\phi^\alpha + GAG^{-1}G\phi^\alpha \,. \tag{90}$$

Inserting another virtual emplacement of G into the first term on the RHS, yields, with (88),

$$\dot{\psi}^\alpha = \dot{G}G^{-1}G\phi^\alpha + GAG^{-1}G\phi^\alpha \,. \tag{91}$$

Thus we finally have

$$\dot{\psi}^\alpha = U\psi^\alpha \,, \tag{92a}$$

where

$$U = \dot{G}G^{-1} + GAG^{-1} = [\dot{G} + GA]G^{-1} \,. \tag{92b}$$

So how do we proceed? We pick a simple G and then substitute it into this last equation. But how do we know ahead of time that the result will make the u_{21} entry of U zero? We don't. We force that by setting $u_{21} = 0$, which is likely to give us another first-order differential equation to solve. Let's hope it's worth all the effort.

So, I used the simple

$$G = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \,, \tag{93a}$$

because it gives us $|G| = 1$ with

$$\dot{G} = \begin{bmatrix} 0 & 0 \\ \dot{x} & 0 \end{bmatrix} \,, \quad G^{-1} = \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix} \,. \tag{93b}$$

[2] A *virtual emplacement* is an operation that leaves the entity unchanged.

As before, we take

$$A = \begin{bmatrix} r_1 & r_2 \\ R_1 & R_2 \end{bmatrix} . \tag{94}$$

Then, on substituting these last values into (92b), we have

$$U = \begin{bmatrix} r_1 - xr_2 & r_2 \\ \dot{x} + xr_1 + R_1 - x(xr_2 + R_2) & xr_2 + R_2 \end{bmatrix} . \tag{95}$$

By the way, notice that in converting A to U, the trace is preserved.

Now we set $u_{21} = \dot{x} + xr_1 + R_1 - x(xr_2 + R_2) = 0$, to obtain

$$U = \begin{bmatrix} r_1 - xr_2 & r_2 \\ 0 & xr_2 + R_2 \end{bmatrix} . \tag{96}$$

with the Riccati equation[3]

$$\dot{x} = q_0 + q_1 x + q_2 x^2 , \tag{97}$$

with $q_0 = -R_1$, $q_1 = R_2 - r_1$, and $q_2 = r_2$. The solution of (97) for $x(t)$ is

$$x(t) = -\dot{y}/q_2 y , \tag{98}$$

where y is the solution to

$$\ddot{y} - R\dot{y} + Sy = 0 , \tag{99}$$

where

$$R = q_1 + \dot{q}_2/q_2 \quad \text{and} \quad S = q_2 q_0 . \tag{100}$$

So, to summarize: If we can solve (99) for y, we can determine U, and then by the method of the last section, determine ψ^α in (86), From that we can recoup ϕ^α from

$$\phi^\alpha = G^{-1}\psi^\alpha . \tag{101}$$

But maybe we can squeeze out a mere algebraic solution to this problem by setting $\dot{x} \equiv 0$, implying that x is a constant. If so, then

$$u_{21} = xr_1 + R_1 - x(xr_2 + R_2) = 0 , \tag{102a}$$

or better yet

$$r_2 x^2 + (R_2 - r_1)x - R_1 = 0 . \tag{102b}$$

Solving, we get

$$x_\pm = \frac{-(R_2 - r_1) \pm \sqrt{(R_2 - r_1)^2 + 4r_2 R_1}}{2r_2} . \tag{103}$$

Furthermore, a necessary condition that these roots be constants is that their product be a constant:

$$x_+ x_- = -\frac{R_1}{r_2} . \tag{104}$$

[3]Taken from Wikipedia.

Eq. (103) looks like a rigorous constraint on the entries of A. How can we make good use of it? The simplest way to simplify the constraint on A is if A is *diapotent*, that is, if all its diagonal entries are the same. In that case, $R_2 - r_1 = 0$ and (103) becomes

$$x_{\pm} = \pm\sqrt{R_1/r_2}\,, \tag{105}$$

where, in general, are all the numbers are complex. In other words, in the matrix A, if R_1/r_2 is a constant, then $\dot{x} = 0$ and we can use

$$U = GAG^{-1} \tag{106}$$

instead of (92b).

A nontrivial diapotent matrix is

$$A = \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix}, \tag{107}$$

where $R_1/r_2 = 1$, a constant. We are free to choose from (105) $x = x_+ = 1$. Thus, for this A we can determine U from (96), yielding

$$U = \begin{bmatrix} t-1 & 1 \\ 0 & t+1 \end{bmatrix}. \tag{108}$$

Thus, the solution for ψ^α is given by

$$\begin{bmatrix} \psi^1(t) \\ \psi^2(t) \end{bmatrix} = M(t) \begin{bmatrix} 1 \\ -t+1 \end{bmatrix} + N(t) \begin{bmatrix} t+1 \\ 0 \end{bmatrix}. \tag{109}$$

And to evaluate $M(t)$ and $N(t)$ in Equations (81)–(84), we make the assignments

$$r_1 = u_{11} = t-1\,, \tag{110a}$$

$$r_2 = u_{12} = 1\,, \tag{110b}$$

$$R_2 = u_{22} = t+1\,. \tag{110c}$$

And when we've finished evaluating $M(t)$ and $N(t)$, we find ϕ^α as

$$\phi^\alpha = G^{-1}\psi^\alpha\,, \tag{111}$$

where

$$G^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}. \tag{112}$$

On performing these evaluations, I get fairly easily that

$$M(t) = \frac{M_0}{t-1} e^{t^2+t}\,, \tag{113a}$$

$$p(t) = (1-t) + \frac{1}{t+1}\,, \tag{113b}$$

$$e^{-\int p(t)dt} = (t+1)e^{\frac{1}{2}t^2 - t}\,, \tag{113c}$$

$$f(t) = \left[1 + \frac{t}{t^2-1}\right] \frac{M_0}{1-t} e^{t^2+t}\,. \tag{113d}$$

This leaves me only one more evaluation to perform:

$$N(t) = e^{-\int p(t)dt}\Big[\int^t e^{\int p(\tau)d\tau} f(\tau)d\tau + C\Big]\,, \tag{114a}$$

where C is a constant. An intermediate result is

$$N(t) = (t+1)e^{\frac{1}{2}t^2-t}\Big\{\int^t \frac{-M_0}{\tau^2-1}e^{\frac{1}{2}\tau^2+2\tau}\big[1+\frac{\tau}{\tau^2-1}\big]d\tau + C\Big\}\,, \tag{114b}$$

8 The Method of Undetermined Coefficients — Special Case Generalized: Traceless Matrices

Similar to the 2×2 diapotent matrices are the 2×2 traceless matrices, particularly those of the $SU(2,\mathbb{C})$ algebra, the Pauli algebra. If the trace of A is zero, then $R_2 = -r_1$, then $r_1 - R_2 = 2r_1$

9 Eigenvalues and Eigenvectors :

Suppose we start with the matrix equation $A\mathbf{x} = \lambda\mathbf{x}$ and ask what the eigenvalues and eigenvectors of the transformation are.

$$A\mathbf{x} = \begin{bmatrix} -r_\alpha- \\ -R_\alpha- \end{bmatrix}[x^\alpha] = \begin{bmatrix} r_1 & r_2 \\ R_1 & R_2 \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \lambda\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}\,. \tag{115}$$

We immediately derive two coupled spinor equations

$$r_\alpha x^\alpha = \lambda x^1\,, \tag{116a}$$

$$R_\alpha x^\alpha = \lambda x^2\,. \tag{116b}$$

Eliminating λ between them, we get

$$x^2 r_\alpha x^\alpha = x^1 R_\alpha x^\alpha\,. \tag{117}$$

Assuming that $x^2 \neq 0$, we divide both sides by $(x^2)^2$ and set $\eta = \dfrac{x^1}{x^2}$ and expand to get the following quadratic in η:

$$R_1\eta^2 + (R_2 - r_1)\eta - r_2 = 0\,, \tag{118}$$

the roots of which are given by the quadratic formula:

$$\eta_\pm = \frac{(r_1 - R_2) \pm \sqrt{(r_1-R_2)^2 + 4r_2R_1}}{2R_1}\,. \tag{119}$$

We can simplify this somewhat by making the substitution: $\xi = \dfrac{(r_1 - R_2)}{2R_1}$, yielding

$$\eta_\pm = \xi \pm \sqrt{\xi^2 + r_2/R_1}\,, \tag{120}$$

where the sign of R_1 is accounted for by the $\pm$ in front of the radical sign. The $\eta_\pm$ are useful numbers because $\begin{bmatrix}\eta_+\\1\end{bmatrix}$ and $\begin{bmatrix}\eta_-\\1\end{bmatrix}$ are eigenvectors corresponding to $\lambda_\pm$ ($x^2 \neq 0$), which we will derive right now. We can solve for λ from (116b). By dividing both sides of (116b) by x^2 we get

$$\lambda_\pm = R_1\eta_\pm + R_2\,. \tag{121}$$

10 Problem 1:

Let's do a simple problem to highlight these formulas. Say we have the matrix

$$A = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}, \tag{122}$$

then $\xi = \dfrac{(r_1 - R_2)}{2R_1} = \dfrac{1-4}{4} = \dfrac{-3}{4}$, which gives us from (120), $\eta_+ = \dfrac{1}{2}$ and $\eta_- = -2$. Therefore,

$$\lambda_+ = 5, \qquad \mathbf{v}_+ = \begin{bmatrix}1/2\\1\end{bmatrix} \mapsto \begin{bmatrix}1\\2\end{bmatrix}, \tag{123a}$$

$$\lambda_- = 0, \qquad \mathbf{v}_- = \begin{bmatrix}-2\\1\end{bmatrix}. \tag{123b}$$

Lemma 2: Suppose we know the eigenvalues and eigenvectors to $A\mathbf{x} = \lambda\mathbf{x}$. How do these values change when we replace A by $A' = A + \rho\mathrm{I}$, for ρ any nonzero scalar and I is the 2×2 identity matrix?

Well, for starters, $\xi' = \xi$ because adding the same thing to both entries on the main diagonal cancel each other out in ξ. Consequently, the eigenvectors are invariant under this change. However, the eigenvalues depend on R_2, which does not cancel out.

$$\lambda'_\pm = R'_1\eta'_\pm + R'_2 = R_1\eta_\pm + (R_2 + \rho) = \lambda_\pm + \rho\,. \tag{124}$$

Lemma 3: *The Eigenvectors of a Symmetric Matrix are Orthogonal.* I could demonstrate this by taking the Euclidean inner product of the two eigenvectors, but there is a simpler way. Note that η_+ is an inverse slope of the eigenvector line corresponding to $\mathbf{v}_+$, etc. So, it's enough to show that $\dfrac{1}{\eta_+\eta_-} = -1$, but even simpler to show that $\eta_+\eta_- = -1$. So, we grab both η's from (120), multiply them together to get

$$\eta_+\eta_- = -\frac{r_2}{R_1}\,. \tag{125}$$

But in a symmetric 2×2 matrix, $r_2 = R_1$, therefore, $\eta_+\eta_- = -1$, as we needed to show.

Although the formula in (125) does not apply to upper-triangular matrices, it does have the look of being somewhat profound.

In the next section we remove the constraints that $x^2 \neq 0$ and $R_1 \neq 0$.

11 Eigenvectors/Spinors Using Isotropic Spinors:

In the last section we derived Eqs. (119) and (121) by use of linear algebra. By contrast, let's redo these equations using isotropic spinors. If we multiply (116a) by r_1 and (116b) by r_2 and add them, we get

$$r_1 r_\alpha \widetilde{x}^\alpha + r_2 R_\alpha \widetilde{x}^\alpha = \lambda r_\alpha \widetilde{x}^\alpha \,, \tag{126}$$

where the tilde means that the vector is an eigenvector. From this we construct the isotrope equation

$$[(r_1 - \lambda) r_\alpha + r_2 R_\alpha] \widetilde{x}^\alpha = 0 \,. \tag{127}$$

We use the Main Heuristic to solve for $\widetilde{x}^\alpha$:

$$\widetilde{x}^\alpha = \kappa [(r_1 - \lambda) r^\alpha + r_2 R^\alpha] \,, \tag{128}$$

where κ is an arbitrary nonzero factor. Now, $\widetilde{x}^\alpha$ is one of an infinite number of eigenvectors to this transformation. So, we could set κ to any nonzero value we like.

We can now obtain the characteristic equation for this transformation by substituting the $\widetilde{x}^\alpha$ from (128) into $A\widetilde{x}^\alpha = \lambda \widetilde{x}^\alpha$ and looking at just the first component, say, to get, once again, the characteristic equation

$$\lambda^2 - T_A \lambda + |A| = 0 \,. \tag{129}$$

Definition: Two $n \times n$ matrices A and B are said to be *similar* if there exists an invertible $n \times n$ matrix M such that

$$A = M^{-1} B M \,. \tag{130}$$

However, it doesn't matter if the inverse matrix appears on the right or left because we can always make the substitution $N = M^{-1}$ and claim for similarity that

$$A = N B N^{-1} \,. \tag{131}$$

The Similarity of A to Diagonal and Nondiagonal Matrices
When a transformation matrix has a complete set of eigenvectors, it can be transformed by a similarity operation into a diagonal matrix whose diagonal entries are the eigenvalues of the matrix. I won't present the details in this

paper. You can consult linear algebra resources for that. But I did make the diagram in Figure 1 to help us visualize this process.

If the matrix does not have a complete set of eigenvectors, we cannot generally find a similarity transformation of it to a diagonal matrix, but instead we can transform the matrix to a form known as the Jordan Normal Form or Jordan Canonical Form. From here on, we'll restrict our attention of A to 2×2 matrices, though the truth the statements made may be in a more general setting as well.

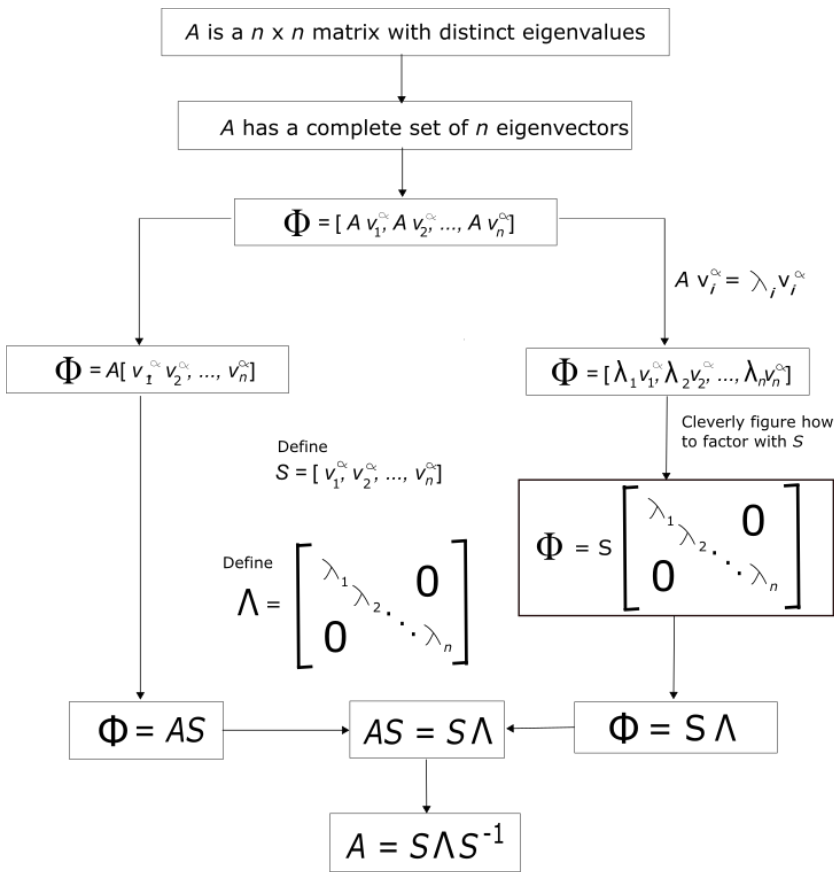

Figure 1. Flowchart of parallel processing of similarity to diagonal matrix. (To reach the final step, we have assumed that the matrix S is invertible.)

If the characteristic equation for a matrix A produces a double root, we have two possibilities: Either the matrix is similar to:

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \tag{132a}$$

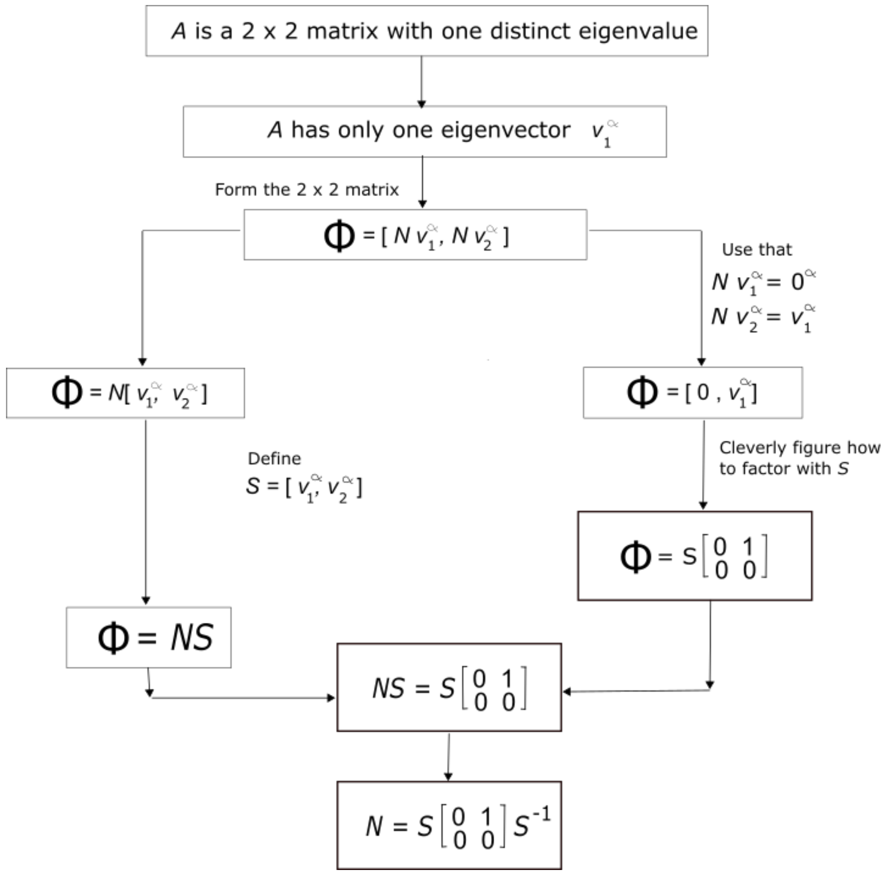

Figure 2. Flowchart of parallel processing of similarity to nondiagonal matrix.

or to:

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} . \tag{132b}$$

Let's first compare to the case when we have two distinct roots as eigenvalues. Then, one at a time we can put these roots into this equation

$$[\lambda \mathbf{I} - A] v_i^\alpha = 0^\alpha \, , \tag{133}$$

to solve for a corresponding eigenvector v_i^α. These eigenvectors are linearly independent and thus can be used to for a transformation matrix

$$S = [v_1^\alpha, v_2^\alpha] \, . \tag{134}$$

(See Figure 1 and/or the literature for a deeper explanation.)

But when we have only one eigenvalue, because of a double root, we can visit (133) only once to get a corresponding eigenvector, say, v_1^α. We ask if there is some way to get a second vector to make a matrix like S in (134) to prove that A is simliar to a Jordan Normal Form? The answer is yes. But this will take some justification, which I'll present now.

First, we form the matrix

$$N = A - \lambda \mathbf{I} . \tag{135}$$

We know that for eigenvector v_1^α

$$N v_1^\alpha = 0^\alpha . \tag{136}$$

Now, very soon we will need to know what the operator N^2 does to vectors in the plane, so let's look at that now. Squaring (135), we get

$$N^2 = A^2 - 2\lambda A + \lambda^2 \mathbf{I} . \tag{137}$$

But according to Cayley-Hamilton Theorem, the RHS of this last equation (with $T_A = 2\lambda$ and $|A| = \lambda^2$) is $\mathbf{0}$. Therefore, first,[4]

$$N^2 = \mathbf{0} , \tag{138}$$

and second,

$$N^2 x^\alpha = 0^\alpha . \tag{139}$$

for all x^α.

The vector v_1^α represents a direction through the origin in the plane, and any vector from the origin to any point on that line (i.e., a scalar multiple of v_1^α), also gets mapped to the zero vector by N. So, what happens to a vector not on this line? Let's see. Choose any nonzero vector u^α not on this line. Then the application of N on this vector must be some other vector in the plane. But since v_1^α and u^α span the plane, we can write the image of u^α under the linear operator N as

$$N u^\alpha = \mu u^\alpha + \tau v_1^\alpha , \tag{140}$$

for some parameters μ and τ. Applying N to both sides of (140), we get, being mindful of (136) as well,

$$N^2 u^\alpha = \mu N u^\alpha = 0^\alpha , \tag{141}$$

where we have invoked (139). Now, we cannot claim that $N u^\alpha = 0^\alpha$ because the null space of N is one dimensional and v_1^α has claimed that honor. So, we're forced to claim that $\mu = 0$, and that means we must rewrite (140) as

$$N u^\alpha = \tau v_1^\alpha , \tag{142}$$

[4]An element is said to be *nilpotent* if, when raised to some positive integral power, it goes to zero.

for some nonzero τ. Therefore, we set $v_2^\alpha = \tau^{-1} u^\alpha$, and we now have our second vector to place in the matrix S, as in Figure 2. The couple of equations we will need are

$$\begin{aligned} N v_1^\alpha &= 0^\alpha \\ N v_2^\alpha &= v_1^\alpha \,. \end{aligned} \tag{143}$$

One more time: Why are we so particular how we choose this v_2^α vector? Because we want to have the representative matrix of the family of all matrices similar to A have a particularly simple form, that being to have only zeros or ones above the main diagonals of each Jordan block in the matrix. See Figure 2 for a demonstration how this works. (Though in higher dimensions, the Jordan forms are, generally speaking, much more complicated.)

So, once we have found the matrix S, using (143), we can write

$$N = S \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S^{-1} \,, \tag{144}$$

and we're finally ready to show the similarity family of the matrix A: Starting with Eq. (135), we can write

$$\begin{aligned} A &= \lambda \mathbf{I} + S \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S^{-1} \\ &= S \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} S^{-1} + S \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S^{-1} \\ &= S \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} S^{-1} \,. \end{aligned} \tag{145}$$

In other words, what we have just shown is that the matrix A is similar to the Jordan Canonical Form:

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \,. \tag{146}$$

12 Coupled First-Order Linear Differential Equations

We begin with the homogeneous equation

$$\dot{\mathbf{Y}} = A \mathbf{Y} \,, \tag{147}$$

where the overdot denotes differentiation by t. Now, theory tells us that Eq. (147) has two linearly independent solutions, even if its characteristic roots are the same (double roots). We'll look at all the common cases: homogeneous equations and nonhomogeneous (or inhomogeneous) equations. Real roots and complex roots. Distinct roots and double roots, this latter case being the most subtle case and the most interesting case, I think.

In matrix form (147) looks like this

$$\begin{bmatrix} \dot{Y}^1 \\ \dot{Y}^2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix} . \tag{148}$$

For consistency with my paper on 2nd-order linear differential operators, I will replace the Y's by ϕ's, though this is not necessary. Also, the top row of the matrix A is replaced by r_α and the second row by R_α. Equation (148) becomes

$$\begin{bmatrix} \dot{\phi}^1 \\ \dot{\phi}^2 \end{bmatrix} = \begin{bmatrix} -r_\alpha- \\ -R_\alpha- \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ R_1 & R_2 \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} = \begin{bmatrix} r_\alpha \phi^\alpha \\ R_\alpha \phi^\alpha \end{bmatrix} , \tag{149}$$

or in short form

$$\dot{\phi}^\alpha = A\phi^\alpha , \tag{150}$$

which is a useful abuse of notation. Or we could be technically correct and tediously write

$$\dot{\phi}^\alpha = A^\alpha_\beta \, \phi^\beta . \tag{151}$$

In this paper, the letters A and B without indices always represent 2×2 matrices.

From (149) we have the coupled spinor equations

$$\dot{\phi}^1 = r_\alpha \phi^\alpha , \tag{152a}$$

$$\dot{\phi}^2 = R_\alpha \phi^\alpha . \tag{152b}$$

Now, if we take r_1 of (152a) and add to it r_2 of (152b), we get

$$r_\alpha \dot{\phi}^\alpha = r_1 r_\alpha \phi^\alpha + r_2 R_\alpha \phi^\alpha . \tag{153a}$$

And if we take R_1 of (152a) and add to it R_2 of (152b), we get

$$R_\alpha \dot{\phi}^\alpha = R_1 r_\alpha \phi^\alpha + R_2 R_\alpha \phi^\alpha . \tag{153b}$$

The rearrangements that finally worked for me were these:

$$r_\alpha [\dot{\phi}^\alpha - r_1 \phi^\alpha] = r_2 R_\alpha \phi^\alpha , \tag{154a}$$

$$R_\alpha [\dot{\phi}^\alpha - R_2 \phi^\alpha] = R_1 r_\alpha \phi^\alpha . \tag{154b}$$

Since we can't cancel any factors, we need to try an ansatz to solve these equations. So, we try

$$\dot{\phi}^\alpha - r_1 \phi^\alpha = C\phi^\alpha , \tag{155a}$$

$$\dot{\phi}^\alpha - R_2 \phi^\alpha = D\phi^\alpha . \tag{155b}$$

where C and D are scalar parameters to be determined. Reconfiguring these last two equations, we have

$$\dot{\phi}^\alpha - (r_1 + C)\phi^\alpha = 0 , \tag{156a}$$

$$\dot{\phi}^\alpha - (R_2 + D)\phi^\alpha = 0 . \tag{156b}$$

From these we get our first needed equation:

$$r_1 + C = R_2 + D\,. \tag{157a}$$

Then, on substituting (155a,155b) into (154a,154b), we get, after a bit of algebra

$$CD = r_2 R_1\,. \tag{157b}$$

Eliminating D between these two equations gives the quadratic

$$C^2 + (r_1 - R_2)C - R_1 r_2 = 0\,, \tag{158}$$

with roots:

$$\begin{aligned} C_\pm &= \frac{-(r_1 - R_2) \pm \sqrt{(r_1 - R_2)^2 + 4r_2 R_1}}{2} \\ &= \frac{(R_2 - r_1) \pm \sqrt{T_A^2 - 4|A|}}{2}\,, \end{aligned}$$

where T_A is the trace of A.

Substituting for $C_\pm$ into (156a) we get, with some algebra,

$$\dot{\phi}^\alpha - \lambda_\pm \phi^\alpha = 0\,, \tag{159}$$

where the eigenvalues are given by

$$\lambda_\pm = \tfrac{1}{2}\big[T_A \pm \sqrt{T_A^2 - 4|A|}\,\big]\,. \tag{160}$$

From this we get the familiar results

$$\lambda_+ + \lambda_- = T_A\,, \tag{161a}$$

$$\lambda_+ \lambda_- = |A|\,, \tag{161b}$$

$$\gamma \equiv \lambda_- - \lambda_+\,, \tag{161c}$$

where γ has been defined for later use.

Equation (159) is the eigenvector equation we've been looking for. Why have we been looking for it in particular? Because we know how to integrate it:

$$\phi^\alpha(t) = c_1 \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda_+ t} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda_- t}\,, \tag{162}$$

where c_1 and c_2 are arbitrary constants for the general solution, and the eigenvectors/spinors $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$ can be determined in a number of ways.

Now, if we substitute $\dot{\phi}^\alpha \mapsto \lambda_\pm \phi^\alpha$ into (153a) and rearrange, we get

$$[(r_1 - \lambda_\pm) r_\alpha + r_2 R_\alpha]\widetilde{\phi}^\alpha = 0\,, \tag{163a}$$

where the tilde over the spinor means that it is an eigenvector/spinor and we have divided out all exponential factors. And if we substitute $\dot{\phi}^\alpha \mapsto \lambda_\pm \phi^\alpha$ into (153b) and rearrange, we get

$$[(r_1 r_\alpha + (r_2 - \lambda_\pm) R_\alpha]\widetilde{\phi}^\alpha = 0\,. \tag{163b}$$

From the Main Heuristic, we can write the eigenvectors in either of the forms

$$\widetilde{\phi}^\alpha_\pm = \kappa_1[(r_1 - \lambda_\pm) r^\alpha + r_2 R^\alpha]\,, \tag{164a}$$

$$\widetilde{\phi}^\alpha_\pm = \kappa_2[r_1 r^\alpha + (r_2 - \lambda_\pm) R^\alpha]\,. \tag{164b}$$

We don't need both of these equations right now. I will use the first of them and set κ_1 to unity, since we are free to scale eigenvectors by any nonzero number we please.

By the way, if the matrix A is symmetric, then, of course $r_2 = R_1$, and the Euclidean inner product of the two eigenvectors $\widetilde{\phi}^\alpha_-$ and $\widetilde{\phi}^\alpha_+$ is zero, that is, that $\widetilde{\phi}^\alpha_- \cdot \widetilde{\phi}^\alpha_+ = [\widetilde{\phi}^\alpha_-]^{\mathrm{T}} \widetilde{\phi}^\alpha_+ = 0$, confirming that the eigenvectors of a symmetric matrix are orthogonal. However, I report that the algebra I used to prove that was significant.

In any case, our equation for the homogeneous solution (162) becomes

$$\phi^\alpha(t) = c_1 \widetilde{\phi}^\alpha_+ \, e^{\lambda_+ t} + c_2 \widetilde{\phi}^\alpha_- \, e^{\lambda_- t}\,, \tag{165}$$

where c_1 and c_2 are arbitrary constants for the general homogeneous solution.

From (150) and (159), we get

$$A\widetilde{\phi}^\alpha_\pm = \lambda_\pm \widetilde{\phi}^\alpha_\pm\,. \tag{166}$$

Because of the isotropic nature of all our spinors, we can write

$$\widetilde{\phi}_\alpha A \, \widetilde{\phi}^\alpha = 0\,, \tag{167}$$

where we have suppressed the $\pm$ on the spinors. This expands to

$$R_1(\widetilde{\phi}^1)^2 + (R_2 - r_1)\widetilde{\phi}^1\widetilde{\phi}^2 - r_2(\widetilde{\phi}^2)^2 = 0\,. \tag{168}$$

With this quadratic form we can rederive (118).

Problem 2

Find the general solution to the problem ([1], p. D-1-21):

$$\dot{\phi}^\alpha(t) = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} \phi^\alpha(t)\,. \tag{169}$$

Using the formulas (120) and (121) we get: $\xi = -\dfrac{1}{8}$ and $\eta_+ = \dfrac{3}{4}, \eta_- = -1$ and

$$\lambda_+ = R_1\eta_+ + R_2 = 6\,, \tag{170a}$$

$$\lambda_- = R_1\eta_- + R_2 = -1\,. \tag{170b}$$

Hence the general solution is

$$\phi^\alpha(t) = c_+ \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{6t} + c_- \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}\,. \tag{171}$$

We can also use formula (160) for $\lambda_\pm$, yielding $\lambda_+ = 6$ and $\lambda_- = -1$ as before. And from the formula (164a) we can get the eigenspinors/vectors. But let's calculate r^α and R^α first.

$$r_\alpha = [2,3] \Rightarrow r^\alpha = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{and} \quad R_\alpha = [4,3] \Rightarrow R^\alpha = \begin{bmatrix} 3 \\ -4 \end{bmatrix}\,. \tag{172}$$

Therefore,

$$\widetilde{\phi}^\alpha_+ = \kappa_1^+\Big((2-6)\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -4 \end{bmatrix}\Big) = \kappa_1^+ \begin{bmatrix} -3 \\ -4 \end{bmatrix}\,. \tag{173}$$

Since we are free to set $\kappa_1^+ = -1$, we get $\widetilde{\phi}^\alpha_+ = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and by a similar calculation we get $\widetilde{\phi}^\alpha_- = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, by which we arrive at the same solution as before.

Problem 3

Find the general solution to the problem ([1], pp. D-1-22-23):

$$\dot{\phi}^\alpha(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \phi^\alpha(t) \quad \text{with initial condition} \quad V^\alpha = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\,. \tag{174}$$

We begin with (160), which gives us $\lambda_\pm = \frac{1}{2}(1 \pm 3)$. Therefore, $\lambda_+ = 2$ and $\lambda_- = -1$. From (164a), we can write

$$r^\alpha = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \quad \text{and} \quad R^\alpha = \begin{bmatrix} -2 \\ -2 \end{bmatrix}\,. \tag{175}$$

$$\widetilde{\phi}^\alpha_+ = (r_1 - \lambda_+) r^\alpha + r_2 R^\alpha = (3-2)\begin{bmatrix} -2 \\ -3 \end{bmatrix} + (-2)\begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\,, \tag{176a}$$

$$\widetilde{\phi}^\alpha_- = (r_1 - \lambda_-) r^\alpha + r_2 R^\alpha = (3-(-1))\begin{bmatrix} -2 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 2 \end{bmatrix}\,. \tag{176b}$$

Therefore, the general solution is

$$\phi^\alpha(t) = c_+ \widetilde{\phi}^\alpha_+ e^{\lambda_+ t} + c_- \widetilde{\phi}^\alpha_- e^{\lambda_- t} = c_+ \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{\lambda_+ t} + c_- \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{\lambda_- t}\,. \tag{177}$$

Now we use the initial condition $V^\alpha = \phi^\alpha(0)$ to solve for the coefficients.

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_+ \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_- \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}\,. \tag{178}$$

It would be easy enough to solve for c_+ and c_- using a matrix inverse or Cramer's Rule, but let's use the isotropy of our spinors instead. Starting with the equation

$$V^\alpha = c_+ \widetilde{\phi}^\alpha_+ + c_- \widetilde{\phi}^\alpha_- \tag{179}$$

and multiply through by $\widetilde{\phi}_{\alpha -}$, we get

$$\widetilde{\phi}_{\alpha -} V^\alpha = c_+ \widetilde{\phi}_{\alpha -} \widetilde{\phi}^\alpha_+ \,. \tag{180}$$

Solving for c_+ we get

$$c_+ = \frac{\widetilde{\phi}_{\alpha -} V^\alpha}{\widetilde{\phi}_{\alpha -} \widetilde{\phi}^\alpha_+}\,, \tag{181a}$$

and similarly for C_-

$$c_- = -\frac{\widetilde{\phi}_{\alpha +} V^\alpha}{\widetilde{\phi}_{\alpha -} \widetilde{\phi}^\alpha_+}\,, \tag{181b}$$

In our particular problem, we get $c_+ = 1$ and $c_- = -1$, yielding the final solution

$$\phi^\alpha(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} e^{-t}\,. \tag{182}$$

See **Appendix 5** for a formula for doing initial value problems.

The Inhomogeneous Equation

The homogeneous equation for a first-order linear differential equation in two variables was given in Eq. (150). We extend the scope of equations we can solve by finding a formula for inhomogeneous equations of the form

$$\dot{\phi}^\alpha = A\phi^\alpha + f^\alpha\,, \tag{183}$$

where f^α is a column spinor, a function of time, but not a function of the ϕ^α variables. The general solution of this equation is the sum of the homogeneous solution $\phi^\alpha_{\rm h}$ and the particular solution $\phi^\alpha_{\rm p}$ that is,

$$\phi^\alpha = \phi^\alpha_{\rm h} + \phi^\alpha_{\rm p}\,. \tag{184}$$

We return to (165) and write it in the simpler form

$$\phi^\alpha(t) = c_1 \phi^\alpha_+ + c_2 \phi^\alpha_-\,, \tag{185}$$

where c_1 and c_2 are arbitrary constants for the general homogeneous solution, and $\phi^\alpha_+ = \widetilde{\phi}^\alpha_+ e^{\lambda_+ t}$ and $\phi^\alpha_- = \widetilde{\phi}^\alpha_- e^{\lambda_- t}$.

To solve for the particular solution we're going to use a technique very similar to that used in the variation of parameters for second-order linear differential equations. We're going to take the parameters in (185), c_1 and c_2, and replace them by explicit functions of the independent variable t. That is, $c_1 \mapsto u_1(t)$ and $c_2 \mapsto u_2(t)$. So our ansatz for (183) becomes

$$\phi^\alpha_{\rm p}(t) = u_1(t)\phi^\alpha_+ + u_2(t)\phi^\alpha_-\,, \tag{186}$$

So, substituting this ansatz into (183), we go in search of a suitable paraquant:

$$\dot{u}_1\phi^\alpha_+ + \dot{u}_2\phi^\alpha_- + u_1\dot{\phi}^\alpha_+ + u_2\dot{\phi}^\alpha_- = A(u_1\phi^\alpha_+ + u_2\phi^\alpha_-) + f^\alpha\,, \tag{187}$$

which can be rearranged to

$$\dot{u}_1\phi^\alpha_+ + \dot{u}_2\phi^\alpha_- + u_1(\dot{\phi}^\alpha_+ - A\phi^\alpha_+) + u_2(\dot{\phi}^\alpha_- - A\phi^\alpha_-) = f^\alpha\,, \tag{188}$$

where the expressions inside the parentheses are zero, leaving us with the paraquant equation

$$\dot{u}_1\phi^\alpha_+ + \dot{u}_2\phi^\alpha_- = f^\alpha\,. \tag{189}$$

One way to proceed here is to form and then solve the standard matrix equation

$$[\phi^\alpha_+, \phi^\alpha_-]\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\end{bmatrix} = \begin{bmatrix}f^1\\ f^2\end{bmatrix}\,. \tag{190}$$

to get

$$\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\end{bmatrix} = [\phi^\alpha_+, \phi^\alpha_-]^{-1}\begin{bmatrix}f^1\\ f^2\end{bmatrix}\,. \tag{191}$$

However, we'll take a different tack. First, let

$$\Phi \equiv \widetilde{\phi}_{-\alpha}\widetilde{\phi}^\alpha_+ = r_2\gamma|A|\,, \tag{192}$$

where we used either (164a) or (164b). Then,

$$H(t) \equiv \phi_{-\alpha}\phi^\alpha_+ = e^{\lambda_+ t}e^{\lambda_- t}\widetilde{\phi}_{-\alpha}\widetilde{\phi}^\alpha_+\,, \tag{193}$$

which simplifies to

$$H(t) = e^{T_A t}\Phi\,. \tag{194}$$

In the special case that the trace of A is zero, $H(t)$ is just the constant Φ.

Now, we proceed by multiplying (189) through by $\phi_{-\alpha}$ to get

$$\dot{u}_1\phi_{-\alpha}\phi^\alpha_+ = \phi_{-\alpha}f^\alpha\,. \tag{195}$$

Solving this for $\dot{u}_1$ and integrating, we get

$$u_1(t) = \int \frac{\phi_{-\alpha}f^\alpha}{H(t)}\,dt = \frac{1}{\Phi}\int e^{(\lambda_- - T_A)t}\widetilde{\phi}_{-\alpha}f^\alpha\,dt\,, \tag{196a}$$

and by a similar process, we get

$$u_2(t) = -\int \frac{\phi_{-\alpha}f^\alpha}{H(t)}\,dt = \frac{-1}{\Phi}\int e^{(\lambda_+ - T_A)t}\widetilde{\phi}_{+\alpha}f^\alpha\,dt\,. \tag{196b}$$

Problem 4

Suppose we have to solve for the general solution to the inhomogeneous equation (barrowed from Dr. Birne Binegar's notes ([2])):

$$\dot{\phi}^\alpha(t) = \begin{bmatrix}2 & -1\\ 3 & -2\end{bmatrix}\phi^\alpha(t) + \begin{bmatrix}e^t\\ e^{-t}\end{bmatrix}\,. \tag{197}$$

The inhomogeneous term as $f^\alpha = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$. Our next task is to find the homogeneous solution. First, we complete the necessary intermediate values: $T_A = 0$, $|A| = -1$,

$$\lambda_+ = 1\,, \quad \lambda_- = -1\,. \tag{198}$$

Without showing the steps, we get

$$\phi^\alpha_{\rm h}(t) = c_1\phi^\alpha_+ + c_2\phi^\alpha_- = c_1 e^t \widetilde{\phi}^\alpha_+ + c_2 e^{-t}\widetilde{\phi}^\alpha_-\,, \tag{199}$$

where

$$\widetilde{\phi}^\alpha_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \widetilde{\phi}^\alpha_- = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\,. \tag{200}$$

And since the trace of A is zero, $H = \Phi = [-3, 1]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2$.

Now, using (196a), we get

$$\begin{aligned} u_1(t) &= \big(-\frac{1}{2}\big)\int e^{-t}[-3,1]\begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}\,dt \\ &= \frac{1}{2}\int \big(3 - e^{-2t}\big)\,dt = \frac{1}{2}\big(3t + \frac{1}{2}e^{-2t}\big)\,, \end{aligned} \tag{201a}$$

and by a similar process we get

$$\begin{aligned} u_2(t) &= -\big(-\frac{1}{2}\big)\int e^{t}[-1,1]\begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}\,dt \\ &= \frac{1}{2}\int \big(1 - e^{2t}\big)\,dt = \frac{1}{2}\big(t - \frac{1}{2}e^{2t}\big)\,. \end{aligned} \tag{201b}$$

Substituting the relevant data into (186), we get for the particular solution

$$\begin{aligned} \phi^\alpha_{\rm p}(t) &= u_1(t)\phi^\alpha_+ + u_2(t)\phi^\alpha_- \\ &= \frac{1}{2}\big(3t + \frac{1}{2}e^{-2t}\big)e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\big(t - \frac{1}{2}e^{2t}\big)e^{-t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \frac{e^t}{4}\Big[\big(6t + e^{-2t}\big)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \big(2te^{-2t} - 1\big)\begin{bmatrix} 1 \\ 3 \end{bmatrix}\Big] \\ &= \frac{e^t}{4}\begin{bmatrix} 6t + e^{-2t} + 2te^{-2t} - 1 \\ 6t + e^{-2t} + 6te^{-2t} - 3 \end{bmatrix}\,. \end{aligned} \tag{202}$$

Therefore, the general solution is

$$\phi^\alpha(t) = \begin{bmatrix} c_1e^t + c_2e^{-t} \\ c_1e^t + 3c_2e^{-t} \end{bmatrix} + \frac{e^t}{4}\begin{bmatrix} 6t + e^{-2t} + 2te^{-2t} - 1 \\ 6t + e^{-2t} + 6te^{-2t} - 3 \end{bmatrix}\,. \tag{203}$$

13 Dealing with Double Roots for λ

When double roots occur in the second-order differential equation with constant coefficients, one uses the trick of taking the first solution $e^{\lambda t}$ and multiplying it by t to get the second solution $te^{\lambda t}$, or the general solution $\phi(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$. (To see the proof of this trick, go to Appendix 6.) There is a similar trick to use when we get a double root to a system of two first-order equations with constant coefficients.

We start with a familiar equation

$$\dot{\phi}^\alpha = A\phi^\alpha \,, \tag{204}$$

and use (160) to get the roots to λ. If the discriminant to this is zero, we have only one root λ. And we can find one eigenvector by using (164a) or (164b), yielding the first solution

$$\phi^\alpha_{(1)} = c_1 e^{\lambda t} \widetilde{\phi}^\alpha \,. \tag{205}$$

However, theory tells us that we have two linearly independent solutions to (204), even with a double root. Someone long ago thought up a clever way to find the second solution to this equation by introducing the ansatz for the second solution

$$\phi^\alpha_{(2)} = c_2 [e^{\lambda t} w^\alpha + t e^{\lambda t} \widetilde{\phi}^\alpha] \,, \tag{206}$$

where w^α is **not** a function of time, and, as before, $A\widetilde{\phi}^\alpha = \lambda\widetilde{\phi}^\alpha$, or equivalently,

$$(A - \lambda \mathrm{I})\, \widetilde{\phi}^\alpha = 0^\alpha = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{207}$$

Our goal is to solve for the undetermined spinor w^α. After we plug our ansatz into (204) and do some cancelling, we get

$$\lambda e^{\lambda t} w^\alpha + e^{\lambda t} \widetilde{\phi}^\alpha = e^{\lambda t} A w^\alpha. \tag{208}$$

And after dividing through by $e^{\lambda t}$ (which is never zero) and do some rearranging, we get the paraquant equation

$$(A - \lambda \mathrm{I}) w^\alpha = \widetilde{\phi}^\alpha. \tag{209}$$

For simplicity, let's substitute $B = A - \lambda \mathrm{I}$ into this to get

$$B w^\alpha = \widetilde{\phi}^\alpha \,, \tag{210}$$

where

$$B = \begin{bmatrix} r_1 - \lambda & r_2 \\ R_1 & R_2 - \lambda \end{bmatrix} = \begin{bmatrix} b_\alpha \\ B_\alpha \end{bmatrix} . \tag{211}$$

Of course $|B| = 0$ because the rows of B are linearly dependent, which means that we can get only one row of useful information out of B at a given calculation. And since B^{-1} doesn't exists, we can't solve for w^α by taking B^{-1} on both sides of (210).

However, the next step in solving for w^α is not hard to grasp. We assume that $w^\alpha = w_h^\alpha + w_p^\alpha$, where w_h^α is the solution to the homogeneous equation

$$Bw_h^\alpha = 0^\alpha \,, \tag{212}$$

which has solution $w_h^\alpha = \mu\widetilde{\phi}^\alpha$, where μ is an arbitrary parameter. That only leaves us with the problem of solving for w_p^α, the particular solution, from the equation

$$Bw_p^\alpha = \widetilde{\phi}^\alpha \,. \tag{213}$$

Let's take a moment out of solving for w_p^α to visualize the geometry involved in the vector w^α. First, we get the homogeneous solution $\mu\widetilde{\phi}^\alpha$, which represents a line through the origin. Then the particular solution w_p^α has the effect of translating that homogeneous solution to another line parallel to itself. Now, it doesn't really matter how that translation is performed so long as all you get the same parallel line. However, the freedom to translate the homogeneous line to the final solution set of points means that w_p^α is not unique.

Okay, remember that (213) corresponds to the two equations

$$b_\alpha w_p^\alpha = \widetilde{\phi}^1 \,, \tag{214}$$

$$B_\alpha w_p^\alpha = \widetilde{\phi}^2 \,, \tag{215}$$

corresponding to the first and second rows of (213). We can use only one of these equations at a time because the second equation will add no additional information. Now, what linear algebra does is to tell us to take one of these equations and tease out some w_p^α – nonuniquely, I might add. But I wish to do better.

I wish to replace the matrix B in (213) with an invertible matrix $\mathcal{B}$ in the adjusted equation

$$\mathcal{B}w_p^\alpha = v^\alpha \,. \tag{216}$$

We'll construct $\mathcal{B}$ by taking one of the equations from (214) or (215) – one with a nonzero $\widetilde{\phi}^\alpha$ component. And we arbitrarily add in an additional constraint on w_p^α. The constraint I'm interested in tells us how to move the origin on the homogeneous solution set of the line through the origin so that the origin 'moves' (or is translated) the shortest distance, and that means that w_p^α is Euclidean-orthogonal to the line through the origin, or that $[\widetilde{\phi}^\alpha]^T w_p^\alpha = 0$.

So, say the first component of $\widetilde{\phi}^\alpha$ is nonzero. Then we form the matrix equation

$$\mathcal{B}w_p^\alpha = \begin{bmatrix} b_\alpha \\ [\widetilde{\phi}^\alpha]^T \end{bmatrix} \begin{bmatrix} w_p^1 \\ w_p^2 \end{bmatrix} = \begin{bmatrix} \widetilde{\phi}^1 \\ 0 \end{bmatrix} . \tag{217}$$

And, if $\mathcal{B}$ is invertible, we solve for w_p^α as

$$w_p^\alpha = \begin{bmatrix} w_p^1 \\ w_p^2 \end{bmatrix} = \begin{bmatrix} b_\alpha \\ [\widetilde{\phi}^\alpha]^T \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\phi}^1 \\ 0 \end{bmatrix} . \tag{218}$$

However, if $\widetilde{\phi}^1 = 0$, we use $\widetilde{\phi}^2$ instead, getting

$$\mathcal{B}w_p^\alpha = \begin{bmatrix} [\widetilde{\phi}^\alpha]^T \\ B_\alpha \end{bmatrix} \begin{bmatrix} w_p^1 \\ w_p^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \widetilde{\phi}^2 \end{bmatrix} . \tag{219}$$

And, once more, if $\mathcal{B}$ is invertible, we solve for w_p^α as

$$w_p^\alpha = \begin{bmatrix} w_p^1 \\ w_p^2 \end{bmatrix} = \begin{bmatrix} [\widetilde{\phi}^\alpha]^T \\ B_\alpha \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \widetilde{\phi}^2 \end{bmatrix} . \tag{220}$$

Having obtained w_p^α, we have the second solution

$$\phi^\alpha_{(2)} = c_2[e^{\lambda t}(w_p^\alpha + \mu\widetilde{\phi}^\alpha) + te^{\lambda t}\widetilde{\phi}^\alpha] . \tag{221}$$

This yields the general solution

$$\phi^\alpha = \phi^\alpha_{(1)} + \phi^\alpha_{(2)} = c_1 e^{\lambda t}\widetilde{\phi}^\alpha + c_2[e^{\lambda t}(w_p^\alpha + \mu\widetilde{\phi}^\alpha) + te^{\lambda t}\widetilde{\phi}^\alpha] , \tag{222}$$

which can be simplified to

$$\phi^\alpha(t) = C_1 e^{\lambda t}\widetilde{\phi}^\alpha + C_2 e^{\lambda t}[w_p^\alpha + t\widetilde{\phi}^\alpha] . \tag{223}$$

Problem 5

Our first example problem is found on pages D-29-30 of [1]. Find the general solution to the initial value problem:

$$\dot{\phi}^\alpha(t) = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \phi^\alpha(t) \quad \text{with} \quad \phi_0^\alpha = \begin{bmatrix} -2 \\ 1 \end{bmatrix} . \tag{224}$$

Using (160) we get the double root $\lambda = -3$. From (164a) we get for the corresponding eigenvector

$$\widetilde{\phi}^\alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \tag{225}$$

with the first solution

$$\phi^\alpha_{(1)} = C_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{226}$$

We solve for w_p^α from (220)

$$w_p^\alpha = \begin{bmatrix} w_p^1 \\ w_p^2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 \\ -1 \end{bmatrix} . \tag{227}$$

Substituting into (223) gives us

$$\phi^\alpha(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t}\Big(\frac{1}{8}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\Big) . \tag{228}$$

Imposing the initial condition, we get

$$\begin{bmatrix}-2\\1\end{bmatrix} = C_1\begin{bmatrix}1\\1\end{bmatrix} + C_2\Big(\frac{1}{8}\begin{bmatrix}1\\-1\end{bmatrix}\Big). \tag{229}$$

Which gives $C_1 = -1/2$ and $C_2 = -12$. Therefore,

$$\phi^\alpha(t) = -\frac{1}{2}e^{-3t}\begin{bmatrix}1\\1\end{bmatrix} - 12e^{-3t}\Big(\frac{1}{8}\begin{bmatrix}1\\-1\end{bmatrix} + t\begin{bmatrix}1\\1\end{bmatrix}\Big) = e^{-3t}\begin{bmatrix}-12t-2\\-12t+1\end{bmatrix}. \tag{230}$$

Problem 6
Our second example problem is found on pages 275-276 of ([3]). Find the general solution to the problem:

$$\dot{\phi}^\alpha(t) = \begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}\phi^\alpha(t). \tag{231}$$

Using (160) we get the double root $\lambda = 1$. From (164a) we get for the corresponding eigenvector

$$\widetilde{\phi}^\alpha = \begin{bmatrix}1\\1\end{bmatrix}, \tag{232}$$

with the first solution

$$\phi^\alpha_{(1)} = C_1 e^t\begin{bmatrix}1\\1\end{bmatrix}. \tag{233}$$

We solve for w^α_p from (220)

$$w^\alpha_p = \begin{bmatrix}w^1_p\\w^2_p\end{bmatrix} = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\end{bmatrix}. \tag{234}$$

Substituting into (223) gives us

$$\phi^\alpha(t) = C_1e^t\begin{bmatrix}1\\1\end{bmatrix} + C_2e^t\Big(\begin{bmatrix}-1/2\\1/2\end{bmatrix} + t\begin{bmatrix}1\\1\end{bmatrix}\Big). \tag{235}$$

The authors's solution is

$$\phi^\alpha(t) = c_1e^t\begin{bmatrix}1\\1\end{bmatrix} + c_2e^t\Big(\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\1\end{bmatrix}\Big). \tag{236}$$

Their solution can be converted to mine by the exchange of parameters: $c_2 \mapsto C_2$ and $c_1 \mapsto C_1 - \frac{1}{2}C_2$.

Problem 7
Our third example problem is found on page 313 of ([4]). Find the general solution to the initial value problem:

$$\dot{\phi}^\alpha(t) = \begin{bmatrix}-2 & 1\\0 & -2\end{bmatrix}\phi^\alpha(t) \quad \text{with} \quad \phi^\alpha_0 = \begin{bmatrix}x_0\\y_0\end{bmatrix}. \tag{237}$$

Using (160) we get the double root $\lambda = -2$. From (164a) we get for the corresponding eigenvector

$$\widetilde{\phi}^{\alpha} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{238}$$

with the first solution

$$\phi^{\alpha}_{(1)} = C_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{239}$$

We solve for w^{α}_{p} from (220)

$$w^{\alpha}_{p} = \begin{bmatrix} w^1_p \\ w^2_p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{240}$$

Substituting into (223) gives us

$$\phi^{\alpha}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-2t} \Big(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Big). \tag{241}$$

Using the initial condition $\phi^{\alpha}_{0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ we can solve for the parameters C_1 and C_2 to get the authors's solution

$$\phi^{\alpha}(t) = e^{-2t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t e^{-2t} \begin{bmatrix} y_0 \\ 0 \end{bmatrix}. \tag{242}$$

Problem 8

Our fourth example problem is found on page 8 of ([5]). Find the general solution to the initial value problem:

$$\dot{\phi}^{\alpha}(t) = \begin{bmatrix} 2 & -3 \\ 3 & 8 \end{bmatrix} \phi^{\alpha}(t) \quad \text{with} \quad \phi^{\alpha}_{0} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \tag{243}$$

Using (160) we get the double root $\lambda = 5$. From (164a) we get for the corresponding eigenvector

$$\widetilde{\phi}^{\alpha} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{244}$$

with the first solution

$$\phi^{\alpha}_{(1)} = C_1 e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{245}$$

We solve for w^{α}_{p} by taking an inverse of $\mathcal{B}$:

$$w^{\alpha}_{p} = \begin{bmatrix} -3 & -3 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -1 \\ -1 \end{bmatrix},. \tag{246}$$

Hence

$$\phi^{\alpha}(t) = C_1 e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{5t} \Big(\frac{1}{6} \begin{bmatrix} -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Big). \tag{247}$$

Using the initial condition $\phi_0^\alpha = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, we can solve for the parameters C_1 and C_2 to get

$$\phi^\alpha(t) = -\frac{1}{2}e^{5t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 15e^{5t}\Big(\frac{1}{6}\begin{bmatrix} -1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}\Big) = e^{5t}\begin{bmatrix} 2-15t \\ 3+15t \end{bmatrix}, \quad (248)$$

which differs from the author's solution.

14 Dealing with Complex Roots for λ

Complex roots for λ pose no particular problem theoretically, though they do add a bit more algebraic manipulation to deal with. The eigenvector/spinors associated with them are complex and the real and imaginary values of the eigenvectors represent independent solutions to the problem. Let's do an example.

Say we have the following problem to solve.

$$\dot{\phi}^\alpha = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\phi^\alpha . \quad (249)$$

From the formula for the eigenvalues (160), we have that

$$\lambda_+ = 1+i \quad \text{and} \quad \lambda_- = 1-i , \quad (250)$$

and for the eigenvectors (164a)

$$\widetilde{\phi}_+^\alpha = \begin{bmatrix} 1-i \\ 1+i \end{bmatrix} \quad \widetilde{\phi}_-^\alpha = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} , \quad (251)$$

where, as usual, we have set the κ factor to unity. And, as usual, our general solution to this problem is given by

$$\phi^\alpha(t) = c_1\widetilde{\phi}_+^\alpha\, e^{\lambda_+ t} + c_2\widetilde{\phi}_-^\alpha\, e^{\lambda_- t} . \quad (252)$$

If we're content to leave our solution in this form, we're done. But if we want our solution to be a real part plus an imaginary part, we've a bit of algebra to do.

First, we factor out the overall factor of e^t:

$$\phi^\alpha(t) = e^t\big[c_1\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}\, e^{it} + c_2\begin{bmatrix} 1+i \\ 1-i \end{bmatrix}\, e^{-it}\big] . \quad (253)$$

Next, we expand the complex exponentials by use of Euler's Formula, to get

$$\begin{aligned} \phi^\alpha(t) &= e^t\big[c_1\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}\,(\cos t + i\sin t) + c_2\begin{bmatrix} 1+i \\ 1-i \end{bmatrix}(\cos t - i\sin t)\big] \\ &= e^t\big[c_1\begin{bmatrix} \cos t + i\sin t - i\cos t + \sin t \\ \cos t + i\sin t + i\cos t - \sin t \end{bmatrix} \\ &\qquad + c_2\begin{bmatrix} \cos t - i\sin t + i\cos t + \sin t \\ \cos t - i\sin t - i\cos t - \sin t \end{bmatrix}\big] . \end{aligned}$$

For the real part of this we get

$$\mathrm{Re}(\phi^\alpha) = e^t\Big[\big(c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix}\big)\cos t + \big(c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix}\big)\sin t\Big]\,. \tag{254}$$

And for the imaginary part of this we get

$$\mathrm{Im}\,(\phi^\alpha) = e^t\Big[\big(c_1\begin{bmatrix}-1\\1\end{bmatrix} - c_2\begin{bmatrix}-1\\1\end{bmatrix}\big)\cos t + \big(c_1\begin{bmatrix}1\\1\end{bmatrix} - c_2\begin{bmatrix}1\\1\end{bmatrix}\big)\sin t\Big]\,. \tag{255}$$

Letting

$$g_1 = c_1 + c_2 \quad \text{and} \quad g_2 = c_1 - c_2\,, \tag{256}$$

we can write the general solution in the following form

$$\phi^\alpha(t) = e^t\Big[g_1\big(\begin{bmatrix}1\\1\end{bmatrix}\cos t + \begin{bmatrix}1\\-1\end{bmatrix}\sin t\big) + ig_2\big(\begin{bmatrix}1\\-1\end{bmatrix}(-\cos t) + \begin{bmatrix}1\\1\end{bmatrix}\sin t\big)\Big]\,. \tag{257}$$

15 Conclusion

The isotropic nature of the spinors we defined has afforded us an alternative to the standard linear algebra treatment of the subject. The symplectic inner product allows us to remove terms by multiplication (with summation) and to hide bulky determinants inside every symplectic inner product. By using the isotropy of the spinors, we are able at times to by-pass solving matrix equations to solve for unknowns. The methods shown here are often more similar to those of vector calculus than linear algebra. We don't often need to talk about null spaces or linear dependence of rows or columns of a matrix And we have formulas for both eigenvalues and eigenvectors to use in solving real problems.

16 Appendix: Spinor Cheat Sheet

Given spinor $A^\alpha \mapsto \begin{bmatrix}A^1\\A^2\end{bmatrix}$, then

$$A_\alpha \mapsto [A_1, A_2] = [-A^2, A^1]\,. \tag{258}$$

Given spinor $B_\alpha \mapsto [B_1, B_2]$, then

$$B^\alpha \mapsto \begin{bmatrix}B_2\\-B_1\end{bmatrix}\,. \tag{259}$$

Given spinor $\widehat{A}^\alpha \mapsto \begin{bmatrix}\bar{A}^1\\A^2\end{bmatrix}$, then

$$\widehat{A}_\alpha \mapsto [A_1, \bar{A}_2]\,. \tag{260}$$

17 Appendix: More on Raising and Lowering of Indices

First, remember that

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{261}$$

Immediately, we can write $J^2 = -\mathrm{I}$, $J^{\mathrm{T}} = -J$, and $J^{\mathrm{T}}J = JJ^{\mathrm{T}} = \mathrm{I}$.

Now to converting between upper and lower indices. Given x^α, we get

$$x_\alpha = (Jx^\alpha)^{\mathrm{T}}\,. \tag{262}$$

Given x_α, we have, using the form given in the last equation

$$[x_\alpha J]^{\mathrm{T}} = [(Jx^\alpha)^{\mathrm{T}}J]^{\mathrm{T}} = J^{\mathrm{T}}(Jx^\alpha) = x^\alpha\,. \tag{263}$$

Theorem
Suppose that A is invertible and we are given that

$$x_\alpha = y_\alpha A\,. \tag{264}$$

Then x^α is given by

$$x^\alpha = A^\dagger y^\alpha\,, \tag{265}$$

where

$$A^\dagger \equiv J^{\mathrm{T}}A^{\mathrm{T}}J\,, \tag{266}$$

the *symplectic conjugate* of A.

Proof: Multiply (264) through by J on the right and replace y_α by its raised equivalent given by (262), and we have

$$x_\alpha J = (Jy^\alpha)^{\mathrm{T}}AJ = (y^\alpha)^{\mathrm{T}}(J^{\mathrm{T}}AJ)\,. \tag{267}$$

Now take the transpose on both sides to get

$$x^\alpha = (J^{\mathrm{T}}A^{\mathrm{T}}J)y^\alpha = A^\dagger y^\alpha\,. \tag{268}$$

By the way, it's not difficult to prove that $(A^\dagger)^\dagger = A$.

18 Appendix: Completion of Inverse of Matrix A

Our next task should be to show that B is also a left inverse, i.e., that $BA = I$. To do this with rows-times-columns method, we wil have to repartition A into columns:

$$A = [t^\alpha, T^\alpha] \quad \text{with} \quad t^\alpha \equiv \begin{bmatrix} r_1 \\ R_1 \end{bmatrix} \quad \text{and} \quad T^\alpha \equiv \begin{bmatrix} r_2 \\ R_2 \end{bmatrix}\,, \tag{269}$$

Now, to perform the multiplication where B is to the left of A we also need to convert the column spinors of B into row spinors, and probably find them as functions of the symplectic transposes of t^α and T^α. Why? Because that's how we're going to get 1's and 0's in the product matrix.

$$t_\alpha = [-R_1, r_1] \quad \text{and} \quad T_\alpha = [-R_2, r_2]\,, \tag{270}$$

Apart from the factor of $\dfrac{1}{|A|}$ for B, B transforms as

$$\begin{aligned} B &\sim [R^\alpha, -r^\alpha] \qquad (271)\\ &= \left[\begin{bmatrix} R^1 \\ R^2 \end{bmatrix}, \begin{bmatrix} -r^1 \\ -\,r^2 \end{bmatrix}\right] = \begin{bmatrix} (R^1, -r^1) \\ (R^2, -r^2) \end{bmatrix} = \begin{bmatrix} (R_2, -r_2) \\ (-R_1, r_1) \end{bmatrix} = \begin{bmatrix} -T_\alpha \\ t_\alpha \end{bmatrix}, \end{aligned}$$

Therefore,

$$BA = \frac{1}{|A|}\begin{bmatrix} -T_\alpha \\ t_\alpha \end{bmatrix}[t^\alpha, T^\alpha] = \frac{1}{|A|}\begin{bmatrix} -T_\alpha t^\alpha & -T_\alpha T^\alpha \\ t_\alpha t^\alpha & t_\alpha T^\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{272}$$

Of course, we can also use the Cayley-Hamilton theorem to solve for A inverse.

19 Appendix: Solving $Ax^\alpha = y^\alpha$ for x^α

Method 1:

Here we assume that A is invertible. Our problem is to solve for x^α in the equation

$$Ax^\alpha = y^\alpha\,. \tag{273}$$

We use the trick of solving an isotrope by multiplying this through by y_α to get

$$y_\alpha A x^\alpha = 0\,. \tag{274}$$

From the Main Heuristic we get that

$$x_\alpha = \kappa y_\alpha A\,. \tag{275}$$

Expanding, we get

$$x_\alpha = \kappa\Big(-\begin{vmatrix} r_1 & y^1 \\ R_1 & y^2 \end{vmatrix}, \begin{vmatrix} y^1 & r_2 \\ y^2 & R_2 \end{vmatrix}\Big). \tag{276}$$

On raising the index, we get

$$x^\alpha = \kappa \begin{bmatrix} \begin{vmatrix} y^1 & r_2 \\ y^2 & R_2 \end{vmatrix} \\ \begin{vmatrix} r_1 & y^1 \\ R_1 & y^2 \end{vmatrix} \end{bmatrix}. \tag{277}$$

We can solve for κ either by multiplying through by r_α and solving the equation $r_\alpha x^\alpha = y^1$ or by solving the complementary equation $R_\alpha x^\alpha = y^2$, to get

$$\kappa = \frac{1}{\begin{vmatrix} r_1 & r_2 \\ R_1 & R_2 \end{vmatrix}} = \frac{1}{|A|}\,. \tag{278}$$

And the result looks like the application of Cramer's Rule:

$$x^\alpha = \begin{bmatrix} \dfrac{1}{|A|}\begin{vmatrix} y^1 & r_2 \\ y^2 & R_2 \end{vmatrix} \\ \dfrac{1}{|A|}\begin{vmatrix} r_1 & y^1 \\ R_1 & y^2 \end{vmatrix} \end{bmatrix}\,. \tag{279}$$

Alternatively, we can solve for κ in (275) sooner. Noting that $r_\alpha x^\alpha = y^1 = -x_\alpha r^\alpha$, multiplying (275) by r^α and summing, we get

$$x_\alpha r^\alpha = \kappa y_\alpha A r^\alpha = -y^1\,. \tag{280}$$

But

$$A r^\alpha = \begin{bmatrix} r_\beta \\ R_\beta \end{bmatrix} r^\beta = \begin{bmatrix} 0 \\ -|A| \end{bmatrix}\,. \tag{281}$$

Therefore, $y_\alpha A r^\alpha = -y_2|A| = -y^1|A|$. Using this in (280), we get

$$\kappa = \frac{1}{|A|}\,. \tag{282}$$

Using this result in (275) we get

$$x_\alpha = \frac{y_\alpha A}{|A|}\,. \tag{283}$$

Now taking the symplectic conjugate of both sides, we get

$$x^\alpha = \frac{A^\dagger y^\alpha}{|A|}\,. \tag{284}$$

Method 2:

Here we take advantage of the work we already did in a previous appendix. Given $Ax^\alpha = y^\alpha$, and

$$x^\alpha = \kappa A^\dagger y^\alpha \tag{285}$$

Multiply this last equation through by A to get

$$Ax^\alpha = \kappa A A^\dagger y^\alpha \tag{286}$$

But the LHS is equal to y^α so we get the matrix equation

$$\kappa A A^\dagger = \mathrm{I}\,. \tag{287}$$

Using

$$A = \begin{bmatrix} r_1 & r_2 \\ R_1 & R_2 \end{bmatrix} \quad \text{and} \quad A^\dagger = \begin{bmatrix} R_2 & -r_2 \\ -R_1 & r_1 \end{bmatrix} . \tag{288}$$

we get

$$\kappa \begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{289}$$

obtaining that $\kappa = \dfrac{1}{|A|}$. But Eq. (287) is true if A is replaced by $A^\dagger$. So we can write

$$A\frac{A^\dagger}{|A|} = \frac{A^\dagger}{|A|}A = \mathrm{I}\,. \tag{290}$$

Therefore, A has an inverse in terms of $A^\dagger$:

$$A^{-1} = \frac{A^\dagger}{|A|}\,. \tag{291}$$

And, of course, $A^\dagger$ is the same as the *adjoint* of linear algebra terminology.

20 Appendix: Formula for Solving the Initial Value Problem

Theorem
Formula for the initial value problem when eigenvalues are distinct: We can build a solution for $\phi^\alpha(t)$ by

$$\phi^\alpha(t) = \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda_+ t} + \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda_- t}\,, \tag{292}$$

where a, b, c, d are to be determined. Taking the derivative of (292) yields

$$\dot{\phi}^\alpha(t) = \lambda_+ \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda_+ t} + \lambda_- \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda_- t} = \begin{bmatrix} r_1 & r_2 \\ R_1 & R_2 \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix}\,, \tag{293}$$

If we are given an initial value of $\phi^\alpha(0) = \begin{bmatrix} V^1 \\ V^2 \end{bmatrix}$, then evaluating these last two equations at $t = 0$ gives us

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} V^1 \\ V^2 \end{bmatrix}\,, \tag{294a}$$

$$\lambda_+ \begin{bmatrix} a \\ b \end{bmatrix} + \lambda_- \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} r_\alpha V^\alpha \\ R_\alpha V^\alpha \end{bmatrix}\,. \tag{294b}$$

These last two equations yield the solutions

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} , \tag{295a}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \gamma^{-1} \begin{bmatrix} \lambda_+ V^1 - r_\alpha V^\alpha \\ \lambda_+ V^2 - R_\alpha V^\alpha \end{bmatrix} . \tag{295b}$$

which applies when the eigenvalues are distinct.

Problem 3 Revisited
Suppose we have the initial value problem to solve:

$$\dot{\phi}^\alpha(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \phi^\alpha(t) \quad \text{with} \quad V^\alpha = \begin{bmatrix} 1 \\ -1 \end{bmatrix} . \tag{296}$$

First, we complete the necessary intermediate values: So, $T_A = 1$, $|A| = -2$, and, from (160), we have that

$$\lambda_+ = 2 , \quad \lambda_- = -1 , \quad \gamma = 3 , \tag{297}$$

yielding so far

$$\phi^\alpha(t) = \begin{bmatrix} a \\ b \end{bmatrix} e^{2t} + \begin{bmatrix} c \\ d \end{bmatrix} e^{-t} , \tag{298}$$

After a bit of algebra we get that $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, making our final solution

$$\phi^\alpha(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} e^{-t} . \tag{299}$$

21 Appendix: Dealing with Double Roots in a Single Second-Order DEQ with Constant Coefficients

We begin with the search for the general solution to the following second-order DEQ with constant coefficients

$$a\ddot{\phi} + b\dot{\phi} + c\phi = 0 . \tag{300}$$

When we try the ansatz to (300) of $\phi = e^{rt}$, we get the equation

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt} = 0 , \tag{301}$$

from which we conclude that

$$ar^2 + br + c = 0 . \tag{302}$$

By the quadratic formula, we get the roots as

$$r_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,. \tag{303}$$

When the discriminant is zero, we get the double root

$$r = \frac{-b}{2a}\,. \tag{304}$$

Therefore, for our first solution to (300) we have

$$\phi_1 = c_1 e^{\frac{-bt}{2a}}\,. \tag{305}$$

The trick to getting the second independent solution to (300) is to try a solution in the form $\phi = u(t)e^{rt}$.[5] When we do, we get, after cancelling the exponential factors,

$$a\ddot{u} + (2ra + b)\dot{u} + (ar^2 + br + c)u = 0\,. \tag{306}$$

Now, the coeffiicient of $\dot{u}$ is zero because of (304) and the coefficient of u is zero because of (302), leaving us with

$$\ddot{u} = 0\,. \tag{307}$$

The most general solution to this is, of course, $u(t) = k_1 t + k_2$. However, we'll drop the arbitrary constants because they will be dealt with when building the general solution to (300), anyway. This leaves us with $u(t) = t$ and the second independent solution to (300) is $\phi_2 = te^{rt}$, with full solution to it being

$$\phi(t) = c_1 e^{rt} + c_2 t e^{rt}\,, \tag{308}$$

where r is given by (304). One can use the Wronskian of e^{rt} and te^{rt} to prove that the two solutions are indeed linearly independent.

22 Appendix: How to Convert a Coupled First-Order Differential System to a Single Second-Order DEQ

This appendix is included for completeness of subject matter.

So, say we wish to convert the following system of first-order differential homogeneous equations into one second-order equation, say in ϕ^2.

$$\dot{\phi}^1 = r_1\phi^1 + r_2\phi^2\,, \tag{309a}$$

$$\dot{\phi}^2 = R_1\phi^1 + R_2\phi^2\,. \tag{309b}$$

[5] Taking $c_1 \mapsto u(t)$ can be considered a form of *variation of parameter*.

How can we do this?

We start by differentiating (309b), to get

$$\ddot{\phi}^2 = R_1\dot{\phi}^1 + R_2\dot{\phi}^2 \,. \tag{310}$$

Next we remove $\dot{\phi}^1$ by using (309a)

$$\begin{aligned}\ddot{\phi}^2 &= R_1[r_1\phi^1 + r_2\phi^2] + R_2\dot{\phi}^2 \\ &= r_1[R_1\phi^1] + R_1 r_2\phi^2 + R_2\dot{\phi}^2 \,.\end{aligned}$$

Next we remove $R_1\phi^1$ by using (309b) and then perform obvious simplifications:

$$\begin{aligned}\ddot{\phi}^2 &= r_1[(\dot{\phi}^2 - R_2\phi^2) + R_1 r_2\phi^2] + R_2\dot{\phi}^2 \\ &= (r_1 + R_2)\dot{\phi}^2 + (-r_1 R_2 + R_1 r_2)\phi^2 \,.\end{aligned}$$

Finally, we get

$$\ddot{\phi}^2 - T_A\dot{\phi}^2 + |A|\phi^2 = 0 \,. \tag{311}$$

And if we try the ansatz $\phi^2 = ce^{\lambda t}$ and substitute this into the above equation and cancel out the removable stuff, we end up with the so-called *characteristic equation*

$$\lambda^2 - T_A\lambda + |A| = 0 \,. \tag{312}$$

This is the same equation one gets by taking the determinant of $\lambda\mathrm{I} - A$ in the usual treatment of the subject.

Let's do the calculation for ϕ^1 by a slightly different method. In compact form, (309a) becomes $\dot{\phi}^1 = r_\alpha\phi^\alpha$. Differentiating this, we get

$$\begin{aligned}\ddot{\phi}^1 &= r_\alpha\dot{\phi}^\alpha = r_1 r_\alpha\phi^\alpha + r_2 R_\alpha\phi^\alpha \\ &= (r_1 r_\alpha + r_2 R_\alpha)\phi^\alpha \\ &= (r_1 r_1 + r_2 R_1)\phi^1 + (r_1 + R_2)(r_2\phi^2) \\ &= (r_1 r_1 + r_2 R_1)\phi^1 + (r_1 + R_2)(\dot{\phi}^1 - r_1\phi^1) \quad \text{(from (309a))} \\ &= T_A\dot{\phi}^1 - |A|\phi^1 \,.\end{aligned}$$

From the theory of 2nd-order differential equations, our second-degree homogeneous equations with constant coefficients have as general solutions, first,

$$\phi^1 = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \,. \tag{313}$$

and second,

$$\phi^2 = c_1' e^{\lambda_1 t} + c_2' e^{\lambda_2 t} \,. \tag{314}$$

Putting these together, we get

$$\begin{bmatrix}\phi^1 \\ \phi^2\end{bmatrix} = e^{\lambda_1 t}\begin{bmatrix}c_1 \\ c_1'\end{bmatrix} + e^{\lambda_2 t}\begin{bmatrix}c_2 \\ c_2'\end{bmatrix} \,. \tag{315}$$

23 Appendix: Determinant of a Matrix Product

Theorem. The determinant of a product of 2×2 matrices is equal to the product of their determinants, or, in terms of matrices A and B:

$$|AB| = |A||B| \,. \tag{316}$$

To facilitate row by column multiplication, let's set

$$A = \begin{bmatrix} r_\alpha \\ R_\alpha \end{bmatrix} \quad \text{and} \quad B = [t^\alpha, T^\alpha] \,, \tag{317}$$

$$AB = \begin{bmatrix} r_\alpha \\ R_\alpha \end{bmatrix} [t^\alpha, T^\alpha] = \begin{bmatrix} r_\alpha t^\alpha & r_\alpha T^\alpha \\ R_\alpha t^\alpha & R_\alpha T^\alpha \end{bmatrix} = \begin{bmatrix} r_\alpha t^\alpha & r_\alpha T^\alpha \\ R_\beta t^\beta & R_\beta T^\beta \end{bmatrix} \,, \tag{318}$$

where $\alpha \mapsto \beta$ on the bottom row to ensure that, when the determinant is taken, that the summations in the terms are taken independently of each other.

Now, we take the determinant of (318):

$$\begin{aligned} |AB| &= r_\alpha t^\alpha R_\beta T^\beta - R_\beta t^\beta r_\alpha T^\alpha \\ &= r_\alpha [R_\beta (t^\alpha T^\beta - t^\beta T^\alpha)] \quad \text{(cancellation when } \alpha = \beta) \\ &= \sum_{\beta \neq \alpha} r_\alpha [R_\beta (t^\alpha T^\beta - t^\beta T^\alpha)] \,. \end{aligned} \tag{319}$$

Therefore, continuing,

$$\begin{aligned} |AB| &= r_1 [R_2 (t^1 T^2 - t^2 T^1)] - r_2 [R_1 (t^2 T^1 - t^1 T^2)] \\ &= r_1 [R^1 (t_2 T^2 + t_1 T^1)] + r_2 [R^2 (t_1 T^1 + t_2 T^2)] \\ &= (r_1 R^1 + r_2 R^2)(t_1 T^1 + t_2 T^2)] \\ &= (r_\alpha R^\alpha)(t_\beta T^\beta) \\ &= |A||B| \,. \end{aligned} \tag{320}$$

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