2nd-order Linear Differential Equations with Isotropic Spinors

P. Reany

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Abstract

I introduced this novel isotropic-spinor method of manipulating linear differential equations back in the 1980s. I re-introduce it here and extend its application to include a treatment of the Sturm-Liouville Problem and Green's Functions. And in the appendices, I reveal some spinors solutions to solving a system of two equations in two unknowns and compare them to a linear and geometric algebra solutions.

1 Introduction

It will help the reader to already know a bit about the subject of 2nd-order linear differential equations up to working with the so-called Wronskian of the system. Though I will try to make the subject as self-contained as possible, I will also try to keep the pace moving. While we're talking about preparation for this paper, it would be helpful for the reader to know about expanding functions as an infinite sum of basis elements, such as is done in Fourier series decomposition of functions by sines and cosines. For a comparison to conventional proofs of the following problems, see [1] or any modern book on ordinary differential equations (ODEs).

Motivating isotropic spinors from the vector cross product:
To motivate the approach here, consider angular momentum in the plane. One of the nicest proofs of classical vector calculus in the realm of physics is the proof that the angular momentum of a point object in a central field has constant angular momentum, regardless of its orbit in space.

Let's start by simplifying our model so that both the object exerting the force and the object feeling the force are point particles. Furthermore, we take as the origin of our coordinate system the center of the source of the force field (fixed in space), and denote it as the point $\mathbf{0}$. Let $\mathbf{r}$ represent the vector from the origin to the instantaneous position of the moving particle. In a central force field, the direction of the force is always along the vector $\mathbf{r}$. Let $\hat{\mathbf{r}}$ be the unit vector representing the direction of $\mathbf{r}$. We represent the vector force on the object by $\mathbf{f}$ and its magnitude by f. Therefore, $\mathbf{f} = \pm\,|\,f\,|\,\hat{\mathbf{r}} = f\hat{\mathbf{r}}$. Now, on to the short proof.

The angular momentum of the particle is defined as the quantity $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p}$ is the linear momentum of the particle in the chosen coordinate system, $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$, where the dot means to differentiate by time. So, we can show that the angular momentum is a constant of the motion if we can show that the time derivative of $\mathbf{L}$ is zero. So,

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times m\dot{\mathbf{r}}) = \dot{\mathbf{r}} \times m\dot{\mathbf{r}} + \mathbf{r} \times m\ddot{\mathbf{r}}\,. \tag{1}$$

But the first term on the right is zero because the vector cross product of any vector with a scalar multiple of itself is zero, $\mathbf{A} \times \alpha\mathbf{A} = \mathbf{0}$. Therefore, we are left with the simpler equation

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times m\ddot{\mathbf{r}}\,. \tag{2}$$

But, since $\mathbf{f} = m\ddot{\mathbf{r}} = f\hat{\mathbf{r}}$, then

$$\frac{d\mathbf{L}}{dt} = r\hat{\mathbf{r}} \times f\hat{\mathbf{r}} = \mathbf{0}\,. \tag{3}$$

Hence, $\mathbf{L}$ does not change in time, making it a constant of the motion. The constancy of $\mathbf{L}$ means that the orbit of the particle will lie in a single plane containing the origin. Our analysis has revealed a 'hidden geometry' of the problem, namely, that the motion is restricted to a fixed plane in space. Beautiful.

Now, let's look at this cross product in 3 dimensions.

$$\begin{aligned}\mathbf{A} \times \mathbf{B} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \end{bmatrix} &= (A^2B^3 - B^2A^3)\mathbf{i} \\ &\quad -(A^1B^3 - B^1A^3)\mathbf{j} + (A^1B^2 - B^2A^1)\mathbf{k}\,,\end{aligned} \tag{4}$$

where we have used superscripts to denote components of a column vector. Soon, we willneed both column vectors and row vectors that use subscripts. Making this distinction is quite necessary in this paper because converting a column vector to a row vector is often more complicated than merely taking a transpose. In Appendix One, I have provided a Spinor Cheatsheet to help keep this distinction straight.

In the angular momentum example above, we had both $\mathbf{r}$ and $\mathbf{p}$ lying in a single plane throughout the course of the motion. We will take rectangular coordinates to mark the orbit of the particle in the xy-plane. Therefore, $\mathbf{r} = (x, y, 0)^t$, where the superscript t means to take the transpose. And $\mathbf{p} = (p_x, p_y, 0)^t$. Therefore, $\mathbf{r} \times \mathbf{p}$ becomes

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ p_x & p_y & 0 \end{bmatrix} = \begin{vmatrix} x & y \\ p_x & p_y \end{vmatrix}\mathbf{k} = (xp_y - yp_x)\mathbf{k} = L_z\mathbf{k}\,. \tag{5}$$

Since the angular momentum is in the direction normal to the plane and will not change in this problem, the only information we need is $L_z = xp_y - yp_x$, effectively just a scalar quantity. If you remember taking the cross product of two vectors **A** and **B** to get the area of a paralellogram, this should seem similar.

$$\mathbf{A} \times \mathbf{B} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A^1 & A^2 & 0 \\ B^1 & B^2 & 0 \end{bmatrix} = \begin{vmatrix} A^1 & A^2 \\ B^1 & B^2 \end{vmatrix} \mathbf{k} = (A^1B^2 - A^2B^1)\mathbf{k} \,. \tag{6}$$

where the quantity $A^1B^2 - A^2B^1$ can be positive or negative. This process of getting a scalar quantity from two vectors is called 'taking a scalar product' on them — which, in the above case, is merely the coefficient of **k**. This can be done in many ways. We are going to define our scalar product so that the result is effectively the determinant of the two vectors in the two dimensions of interest. To that end, we start with any two vectors **A** and **B** in 2-space and represent them in matrix form as $\begin{pmatrix} A^1 \\ A^2 \end{pmatrix}$ and $\begin{pmatrix} B^1 \\ B^2 \end{pmatrix}$, respectively. These vectors also have a convenient component, or indicial, form as A^α and B^α, respectively, where α takes on values 1,2.

Now, you're probably familiar with the Euclidean inner product of two vectors, but we aren't going to use that inner product. (If we're going to capture the information in a cross product as a 'scalar product', this makes sense to use some other notion of inner product.) But for comparison's sake, we'll show what the Euclidean inner product of two vectors B^α, A^α looks like:

$$B_\alpha A^\alpha \equiv \sum_\alpha B_\alpha A^\alpha = B_1A^1 + B_2A^2 \,. \tag{7}$$

In other words, to scalar-multiply two column vectors together, we will have to convert one of them to a row vector by this procedure: $B_\alpha = [J^t B^\alpha]^t = (B^\alpha)^t J$, where J is a 2×2 matrix used to lower the index of a vector. For the Euclidean case, $J = I$ is the 2×2 identity matrix, and

$$B_\alpha \mapsto (B_1, B_2) \equiv (B^1, B^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (B^1, B^2) \,, \tag{8}$$

with the resulting familiar Euclidean square

$$B_\alpha B^\alpha = (B^1, B^2) \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} = (B^1)^2 + (B^2)^2 \,. \tag{9}$$

However, for our symplectic inner product, we take the **symplectic matrix** $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and get, with

$$B_\alpha \equiv (JB^\alpha)^{\mathrm{T}} = B^{\alpha \mathrm{T}} J^{\mathrm{T}} \,, \tag{10}$$

to get

$$B_\alpha \mapsto (B_1, B_2) \equiv (B^1, B^2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (B^2, -B^1) \,. \tag{11}$$

Now we're ready to take the sympectic 'inner' product of vectors B^α and A^α:

$$B_\alpha A^\alpha = (B_1, B_2)\begin{pmatrix}A^1\\A^2\end{pmatrix} = (B^2, -B^1)\begin{pmatrix}A^1\\A^2\end{pmatrix} = B^2A^1 - B^1A^2 \tag{12}$$

And, on taking this inner product in the opposite order, we get

$$A_\alpha B^\alpha = A^2B^1 - A^1B^2 = -(B^2A^1 - B^1A^2) \tag{13}$$

Comparing (12) to (13), we see that they are negatives of each other

$$A_\alpha B^\alpha = -B_\alpha A^\alpha \tag{14}$$

So our symplectic inner product acts like the antisymmetric cross product of two vectors.

Definition: So far the components of our two-component vectors have been real numbers or real-valued functions. But now we let them be complex-valued, in which case we refer to them as **spinors**. Spinors were introduced to physics by Paul Ehrenfest and Wolfgang Pauli in the 1920s to deal with the then novel notion of electron spin. We will not need that particular interpretation of them. Our use of them here is quite formal, but convenient, as we shall see.

Important Lemma
Let A^α and B^α be two spinors. Then their symplectic 'inner' product is

$$A^\alpha B_\alpha = \det\begin{bmatrix}A^1 & A^2\\B^1 & B^2\end{bmatrix} = A^1B^2 - B^1A^2\,. \tag{15}$$

But what happens if we use the same vector in this scalar product? Let's try it. $A_\alpha A^\alpha = -A^2A^1 + A^1A^2 = 0$. And this is true whatever the components of A^α are! This is certainly not what we'd get using a Euclidean inner product, but we want the product to represent a cross product, not a Euclidean length squared.

Definition: A nonzero vector/spinor whose 'square' is zero is said to be **isotropic**.

Definition: If the product of two nonzero vectors/spinors is zero, we shall refer to the product as an **isotrope**.

2 Interlude for some basic theory from ODEs

In the theory of ordinary differential equations (ODEs), every homogeneous linear second-order differential equation has two linearly independent solutions, say y_1 and y_2.

By linearity of the solutions, we mean that any linear combination of these two solutions is also a solution. So, for arbitrary complex numbers c_1 and c_2,

$y_{\text{homo}} = c_1 y_1 + c_2 y_2$ is also a solution to the equation. Let's take the following equation as our standard second-order homogeneous differential equation

$$y'' + a_1(x)y' + a_2(x)y = 0\,. \tag{16}$$

Hence, if y_1 and y_2 are solutions to (16), then $c_1 y_1 + c_2 y_2$ is also a solution.

Let's rewrite (16) in the form of a linear operator L acting on y, where

$$L \equiv D_x^2 + a_1(x)D_x + a_2(x)\,. \tag{17}$$

Then (16) becomes

$$L(y) = 0\,. \tag{18}$$

While we're at it, let's show that $L(y_{\text{homo}}) = 0$.

$$L(y_{\text{homo}}) = L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) = 0\,, \tag{19}$$

where we have used that L is linear and that $L(y_1) = L(y_2) = 0$.

For the time being, we'll take the following equation as our standard nonhomogeneous 2nd-order linear differential equation for y as a function of x, with nonhomogeneous term $b(x)$.

$$y'' + a_1(x)y' + a_2(x)y = b(x)\,, \tag{20}$$

which can also be written in linear operator form, using the same L as defined in (17).

$$L(y) = b(x)\,. \tag{21}$$

The theory of ODEs tell us that if $b(x)$ is nonzero then (21) has a solution, called the *particular* solution, independent of the two homogeneous solutions, and we'll refer to it as y_p. Thus (21) has the complete solution, say $Y = c_1 y_1 + c_2 y_2 + y_p$.

Easy first attempt converting to spinor form

We now go back to Eq. (19) to recast the solutions y_1 and y_2 into spinor form ϕ^α by: $y_1 \mapsto \phi^1$ and $y_2 \mapsto \phi^2$. We introduce the constant spinor $c_\alpha(\alpha = 1, 2)$ to correspond to (c_1, c_2). (By 'constant' I mean that the components of c_α are independent of x but otherwise arbitrary.) Therefore, $y_{\text{homo}} = c_\alpha \phi^\alpha$, where we have used the convention that whenever an index in one term (in this case α) is both upper and lower then we sum on the full range of the index. And this will be our convention throughout the rest of the paper. Now, we can redo (19) this way, taking note that $L(\phi^\alpha) = 0$ for $\alpha = 1, 2$:

$$L(y_{\text{homo}}) = L(c_\alpha \phi^\alpha) = c_\alpha L(\phi^\alpha) = 0\,. \tag{22}$$

Comparing (19) with (22), it's either *Beautiful!* or *Big Deal!* Well, we've yet to see how it all comes together to produce compact and beautiful proofs.

Continuing, the Wronskian can be written as

$$W \equiv \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{vmatrix} \phi^1 & \phi^2 \\ \phi^{1\prime} & \phi^{2\prime} \end{vmatrix} = \phi^\alpha {\phi_\alpha}' \quad \text{(again, summing on } \alpha\text{)}. \tag{23}$$

Since the Wronskian W plays a central role in the theory of ODEs, any simplification working with it should make the theory more efficient. Note that ${\phi_\alpha}'\phi^{\alpha\prime} = 0$ because, $\phi^{\alpha\prime}$ is, like all our spinors, isotropic. By the way, our comparison of the Wronskian to the angular momentum vector seems to hint at some hidden 'symplectic geometry' of the Wronskian.

Some useful lemmas

Lemma 1a: Let the spinors A_α and B^α ($\alpha = 1, 2$) be two functions of x, say. Let $P(x) = A_\alpha B^\alpha$. Then $P' = {A_\alpha}' B^\alpha + A_\alpha B^{\alpha\prime}$. Thus the product rule of differentiation applies to the product of spinors in summation. The proof is simply demonstrated by first expanding the summation in terms of components and then applying the differentiation.

Lemma 1b: Let the spinor $B^\alpha = A^{\alpha\prime}$ in Lemma 1a, then $P(x) = A_\alpha A^{\alpha\prime}$. Therefore $P' = {A_\alpha}' A^{\alpha\prime} + A_\alpha A^{\alpha\prime\prime} = A_\alpha A^{\alpha\prime\prime}$.

Lemma 1c: Let's apply the last lemma to get the derivative of the Wronskian: $W' = \phi_\alpha \phi^{\alpha\prime\prime}$, which is similar to what happened in Eq. (2).

Lemma 1d: Show that ${\phi_\alpha}'\phi^\alpha = -\phi_\alpha\phi^{\alpha\prime}$ for arbitrary spinor ϕ^α. Since $\phi_\alpha\phi^\alpha = 0$, differentiating this gives: ${\phi_\alpha}'\phi^\alpha + \phi_\alpha\phi^{\alpha\prime} = 0$. From this we get that ${\phi_\alpha}'\phi^\alpha = -\phi_\alpha\phi^{\alpha\prime}$.

Lemma 1e: Show that if the Wronskian is zero[1], then ϕ^2 is a constant multiple of ϕ^1 .

$$W = \phi^1\phi^{2\prime} - \phi^2\phi^{1\prime} = 0. \tag{24}$$

Divide both sides by $\phi^1\phi^2 (\neq 0)$ to get

$$\frac{\phi^{2\prime}}{\phi^2} = \frac{\phi^{1\prime}}{\phi^1} \tag{25}$$

or,

$$D_x \ln \phi^2 = D_x \ln \phi^1 \tag{26}$$

or,

$$D_x \ln \frac{\phi^2}{\phi^1} = 0\,. \tag{27}$$

On integrating, we get, for some constant c,

$$\ln \frac{\phi^2}{\phi^1} = c\,. \tag{28}$$

So, for some constant $\kappa = e^c$,

$$\phi^2 = \kappa\phi^1\,. \tag{29}$$

Now we begin to demonstrate the usefulness of the Wronskian in this subject. And our spinor formulation of this subject makes proving all this a bit easier than the conventional approach.

[1] And ϕ^1 and ϕ^2 are analytic functions.

3 Some easy problems to begin with

Problem 1: Show that the Wronskian is constant for every differential equation of the form

$$y'' + q(x)y = 0\,, \tag{30}$$

where y is a function of x. First, we cast our solutions into the spinor form ϕ^α, satisfying

$$\phi^{\alpha\prime\prime} + q(x)\phi^\alpha = 0 \quad (\alpha = 1, 2)\,. \tag{31}$$

And here's where we use the isotropic nature of the spinor ϕ^α. We multiply the above equation by ϕ_α and perform the summation, causing the second term on the left to drop out, leaving us with

$$\phi_\alpha \phi^{\alpha\prime\prime} = -W' = 0\,. \tag{32}$$

From this we conclude that the Wronskian is a constant.

That happened rather quickly. Did we cheat somehow? No. What we did was to design the 'inner' product on our spinors to perform the tedious algebra that we would have had to perform by solving the following coupled system by conventional algebraic mess (Oh, I mean algebraic *means*, of course!):

$$\phi^{1\prime\prime} + q(x)\phi^1 = 0\,, \quad \phi^{2\prime\prime} + q(x)\phi^2 = 0\,. \tag{33}$$

Here's another way to look at it: The indicial equation (31) has all the information that Equations (33) have. The spinor method is merely the short way to eliminate the undifferentiated terms in the coupled equations in (33), which is what most people do to solve this problem, using standard algebra.

Problem 2: Show that the Wronskian for any differential equation of the form

$$y'' + a_1(x)y' + a_2(x)y = 0\,, \tag{34}$$

satisfies the differential equation

$$W' + a_1(x)W = 0 \tag{35}$$

and solve the equation for W. First, we cast our solutions in the form of the spinor ϕ^α:

$$\phi^{\alpha\prime\prime} + a_1(x)\phi^{\alpha\prime} + a_2(x)\phi^\alpha = 0 \quad (\alpha = 1, 2)\,. \tag{36}$$

Then, taking our clue from the last problem, we multiply through by ϕ_α and sum, leaving us with (35) when the spinor dust clears.

From the theory of first-order linear equations, we know that W in (35) has the solution (known as *Abel's formula*)

$$W(x) = C_0 \exp\left\{-\int_{x_0}^{x} a_1(x)dx\right\}\,, \tag{37}$$

where C_0 is an arbitrary nonzero constant.

Problem 3: For the differential equation given in (34), if one solution is known, say $\phi^1(x)$, find a formula for the other solution. In the theory of ODEs this is referred to as a *reduction of order technique*. This solution is presented in Appendix 3.

4 An initial value problem

Say we are given a second-order linear differential homogeneous equation, such as (34), with initial conditions: $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Can we determine the exact values of the c_i's in $y_h = c_1\phi^1 + c_2\phi^2$? Yes, we can. Consider the following pair of equations:

$$c_\alpha \phi^\alpha(x_0) = y_0\,, \tag{38a}$$

$$c_\alpha \phi^{\alpha\prime}(x_0) = y_0'\,. \tag{38b}$$

Note that these equations can be considered either as spinor equations or standard algebra equations. Ignoring technicalities better left for textbooks, so long as the Wronskian at x_0, $W_0 \equiv \phi^\alpha(x_0)\phi_\alpha{}'(x_0)$, is nonzero, we can easily find the c_α by used of Cramer's Rule, for one. But we will use a spinor approach, based on the solution form found in Appendix 2, Method 3, except, instead of using an end-result formula, we will put in all the steps. We try the ansatz

$$c_\alpha = A\phi_\alpha(x_0) + B\phi_\alpha{}'(x_0)\,, \tag{39}$$

where A and B are complex numbers to be determined from the initial values.

$$y_0 = c_\alpha \phi^\alpha(x_0) = BW_0\,, \tag{40a}$$

$$y_0' = c_\alpha \phi^{\alpha\prime}(x_0) = -AW_0\,. \tag{40b}$$

where the minus sign came from the fact that since $\phi_\alpha\phi^{\alpha\prime} = -W$, and $\phi^\alpha\phi_\alpha{}' = W$. Now, solving the above for A and B and substituting these into (39) yields

$$c_\alpha = \frac{1}{W_0}[-y_0'\phi_\alpha(x_0) + y_0\phi_\alpha{}'(x_0)]\,, \tag{41}$$

for $\alpha = 1, 2$, and that's the solution. But to show conformance of this solution to the solution of (38a, 38b) reached by Cramer's Rule, we convert to column spinors. Specifically, first

$$c_1 = \frac{1}{W_0}[-y_0'\phi_1(x_0) + y_0\phi_1{}'(x_0)]\,, \tag{42a}$$

$$c_2 = \frac{1}{W_0}[-y_0'\phi_2(x_0) + y_0\phi_2{}'(x_0)]\,. \tag{42b}$$

Now we use the conversions in Appendix 1:

$$c_1 = \frac{1}{W_0}[-y_0'\phi^2(x_0) + y_0\phi^{2\prime}(x_0)]\,, \tag{43a}$$

$$c_2 = \frac{1}{W_0}[y_0'\phi^1(x_0) - y_0\phi^{1\prime}(x_0)]\,. \tag{43b}$$

Alternatively, we could solve for c_α by constructing the isotrope from (38a, 38b):[2]

$$c_\alpha \left[\phi^\alpha(x_0) y_0' - \phi^{\alpha\prime}(x_0) y_0 \right] = 0 \,, \tag{44}$$

and use the technique described in the next section to solve for c_α.

5 The Main Heuristic

Let X^α and Y^α be nonzero spinors forming the isotrope $Y_\alpha X^\alpha = 0$; then we know that there exists a nonzero scale factor κ, say, such that

$$Y_\alpha = \kappa X_\alpha \,. \tag{45}$$

Say we wish to solve for κ. One way to do this is to have a third spinor Z^α satisfiying two equations

$$Y_\alpha Z^\alpha = a \,, \tag{46a}$$

$$X_\alpha Z^\alpha = b \,. \tag{46b}$$

Then we multiply (45) through by Z^α and sum, yielding the relation $a = \kappa b$ to solve for κ.

We shall see that in solving real problems, either an isotrope will arise naturally in a given problem, or we shall benefit by finding a way to construct one, no matter how artificial that construction may appear, just as the construction of Equation (44) might appear artificial. Let's solve for c_α in (44). The next step is to assert that there exists a κ such that

$$c_\alpha = \kappa [\, \phi_\alpha(x_0) y_0' - {\phi_\alpha}'(x_0) y_0 \,] \,. \tag{47}$$

Let's choose $\phi^\alpha(x_0)$ from (38a) as the third spinor, so, on multiplying (47) through by ϕ^α, we get

$$y_0 = c_\alpha \phi^\alpha(x_0) = -\kappa {\phi_\alpha}'(x_0) \phi^\alpha(x_0) y_0 \,. \tag{48}$$

After simplifying this, we get, $\kappa = \dfrac{-1}{W_0}$, resulting in the same answer for c_α as in (41).

6 The Esteemed Variation of Parameters Solution to the Particular Solution

We take the following as our general nonhomogeneous equation to solve for the particular solution:

$$y'' + a_1(x) y' + a_2(x) y = b(x) \,. \tag{49}$$

[2] Multiply (38a) by y_0' and (38b) by y_0, and then subtract the second altered equation from the first.

The problem posed to us is to find the particular solution y_p to (49) if we already know the solutions to the homogeneous equation. Again, we adopt the linear operator symbol $L = D_x^2 + a_1(x)D_x + a_2(x)$. We have the homogeneous equation companion to (49)

$$L(y) = 0\,, \tag{50}$$

with two linearly independent solutions y_1 and y_2. As we saw before, arbitrary linear combinations of these solutions are also solutions to (50), and the general homogeneous solution is $y_h = c_1 y_1 + c_2 y_2$ with $L(y_h) = 0$.

The audacious plan, formulated long ago, was to seek the particular solution to

$$L(y_p) = b(x) \tag{51}$$

by converting the arbitrary constants in the homogeneous solution to functions of x given by $c_\alpha \mapsto u_\alpha(x)$ where $(\alpha = 1, 2)$. In a manner of speaking, our converting the arbitrary parameters c_α to variables of x is 'varying the parameters'. These new variable parameters u_α will have to be determined, of course. And just how that is accomplished is where the beauty of this proof lies. So, in formal terms:

$$y_p = u_1 y_1 + u_2 y_2\,, \tag{52}$$

which we'll convert into our spinor-form ansatz as

$$y_p = u_\alpha \phi^\alpha\,. \tag{53}$$

It's well at this point to attempt to anticipate the effect of making the substitution of (53) into (51). Equation (51) is second-order in one variable y_p. After substituting (53) into it, we will have to solve for what? Perhaps two coupled equations of first-order in u_α? That makes sense. Let's try it.

$$L(y_p) = L(u_\alpha \phi^\alpha) = u''_\alpha \phi^\alpha + 2{u_\alpha}'\phi^{\alpha\prime} + a_1(x){u_\alpha}'\phi^\alpha + u_\alpha L(\phi^\alpha) = b(x)\,. \tag{54}$$

But, since $L(\phi^\alpha) = 0$ for each $\alpha = 1, 2$ then $u_\alpha L(\phi^\alpha) = 0$ as well. Therefore (54) reduces to

$$u''_\alpha \phi^\alpha + 2{u_\alpha}'\phi^{\alpha\prime} + a_1(x){u_\alpha}'\phi^\alpha = b(x)\,. \tag{55}$$

This is our first constraint equation on u_α, but it's way too complicated. (It's well to remember that the ϕ^α are known functions of x.)

Now it's time to set a judicious constraint on the remaining terms of the LHS of (55) to derive a system of two coupled equations in ${u_\alpha}'(x)$. An obvious choice is to construct the isotrope

$${u_\alpha}'\phi^\alpha = 0\,. \tag{56}$$

But why is this choice "obvious"? First, because it *is* an isotrope! Second, because it is a simple algebraic relation on the functions ${u_\alpha}'$, and, third, because it has the effect of eliminating the term with $a_1(x)$, which should simplify things.[3]

[3]It may seem like something is wrong here to drop the term having $a_1(x)$ in it. Aren't we losing information? Not necessarily, because $a_1(x)$ has already been used to get the homogeneous solutions, which we are representing as ϕ^α.

As a bonus, we can differentiate (56) to get

$$u''_\alpha \phi^\alpha + u_\alpha{}' \phi^{\alpha\prime} = 0\,. \tag{57}$$

Using this last equation and (56) in (55), gives the resulting winnowed-down equation

$$u_\alpha{}' \phi^{\alpha\prime} = b(x)\,. \tag{58}$$

Therefore, (56) and (58) are our sought-after system of first-order equations to solve for u_α. Since (56) is an isotrope, then, following the advice given in the Main Heuristic, we assume that

$$u_\alpha{}' = \kappa(x)\phi_\alpha\,, \tag{59}$$

where $\kappa(x)$ is some function to be determined. Multiplying both sides by $\phi^{\alpha\prime}$ and summing gives the result

$$\kappa(x) = -b(x)/W(x)\,, \tag{60}$$

which we can plug into (59) and integrate, yielding

$$u_\alpha(x) = -\int^x \frac{b(x)\phi_\alpha(x)}{W(x)}\,dx\,. \tag{61}$$

Finally, the solution for y_p is

$$y_p(x) = -\int^x \frac{b(\tau)\phi_\alpha(\tau)\phi^\alpha(x)}{W(\tau)}\,d\tau = \int^x \frac{b(\tau)\phi^\alpha(\tau)\phi_\alpha(x)}{W(\tau)}\,d\tau\,. \tag{62}$$

The standard approaches to this problem do not introduce the function $\kappa(x)$ as seen in Equation (59) because they use linear algebra instead to solve (56) and (58) directly for $u_\alpha{}'$. That approach is straightforward but a tad messier.

By the way, we'll return to this particular solution when we encounter the Green's function concept near the end of this article.

Follow-up discussion on the proofs to The Variation of Parameters

As I compare and contrast the above spinor solution to the Variation of Parameters problem to the standard approaches based on linear algebra, I think that the spinor approach is prettier and its equations far more compact and thus more comprehensible. Either way, the heart of the entire solution lies in coming up with that pair of coupled equations in $u_\alpha{}'$.

Generally speaking, the earlier the constraint (56) is introduced to the reader in textbooks, the less motivated it appears, but the greater its effect on shortening the solution, which is probably fine to most students, who aren't math majors and aren't much interested in proofs anyway. By contrast, I was determined not to introduce the second constraint (56) until after I presented equation (55) to the reader, so I could explain why.

For a few standard treatments of the Variation of Parameters, see ([2]), ([3] and [4]).

Definition: Integration by Parts vs. Contraflux.
Way back in the 1980s, when I was studying tensor calculus, I noticed that authors were using identities such as $A'B = (AB)' - AB'$ and referring to them as 'integration by parts', even though no integral sign was within pages of this identity. Perhaps the authors used that referent for lack of anything else to refer to them by. So, long ago I invented the term *contraflux* to refer to this identity. Here's my explanation: Newton called his derivative a 'fluxion', which I take to be $(AB)'$. Thus, the 'partial flux' $A'B$ of AB is equal to the full flux $(AB)'$ plus a contrary part $-AB'$, hence, contraflux. I shall have occasion to use the contraflux outside of an integrand right away. When contrafluxing occurs in an integrand, it then makes sense to refer to it as integration by parts.

Definition: Paraquant.
I'm here introducing the term **paraquant**. A *paraquant* is an equation constructed similar to a given equation to solve, which, if it can be solved, provides information useful to solve the given equation. Example: Given the original equation $L(y) = f(x)$, where L is a linear operator, the **method of Green's functions** attempts to solve the paraquant equation $L[G(x,\xi)] = \delta(x-\xi)$, to get $y = \int G(x,\xi)f(\xi)d\xi$.

Lemma 2: For a differential equation of the form

$$(py')' + qy' = 0\,, \tag{63}$$

where p and q are functions of x, show that pW is a constant.

Proof:
We begin by converting to spinor form:

$$(p\phi^{\alpha\prime})' + q\phi^{\alpha} = 0\,. \tag{64}$$

Next, we multiply through by ϕ_α, and sum, getting

$$\phi_\alpha(p\phi^{\alpha\prime})' = 0\,. \tag{65}$$

Now, contraflux the LHS and discard the contra term:

$$(p\phi_\alpha\phi^{\alpha\prime})' = (-pW)' = 0\,. \tag{66}$$

Integrating gives us $pW =$ constant.

7 The Sturm-Liouville (S-L) Problem

Part I: Motivation
At this point the level of sophistication jumps dramatically to deal with notions of self-adjoint linear operators, complete sets of basis vectors, and the like. The first thing to clear up is what S-L theory does and does not give us. It does **not** give us a general algorithm to solve for the general solutions to the second-order linear differential equation. What it does do is to provide (at least) four facts:

1. the eigenfunctions to the S-L equation are mutually orthogonal.
2. the eigenvalues corresponding to the eigenfunctions are real.
3. the eigenfunctions to the S-L equation form a 'complete set' .
4. the eigenvalues can be ordered into an unbounded increasing sequence.

Of these four, only the first two are relevant to this paper because they are amenable to proof using the spinor approach advocated here. This notion of a set of functions being a 'complete set' means that any piecewise continuous function can be expanded in terms of them. We will use this fact when using the S-L equation to solve for the particular solution.

We'll begin our investigation of this method of dealing with second-order linear differential equations of the S-L type by introducing the simple version

$$L(y) = \lambda r(x) y \,, \tag{67}$$

where L is a linear operator on the function $y(x)$ (with domain $[a, b]$), which we'll deal with in more detail later. The function $r(x)$ is called the *weight function* and it allows us some flexibility in fashioning suitable problems to solve. As I understand it, the so-called "Sturm-Liouville Problem" amounts to solving for the appropriate λ for a given 'Sturm-Liouville equation.

For now, we notice that this form of equation has immediate relation to such equations of physics as the time-independent Schrödinger equation

$$L(\psi) = -\frac{\hbar^2}{2m} D_x^2 \psi + V(x)\psi = E\psi \,, \tag{68}$$

where $L = -\dfrac{\hbar^2}{2m} D_x^2 + V(x)$ and E takes the role of the eigenvalues to the eigenvectors ψ. Hermite, Bessel, and Legendre equations can also be fitted to this format.

The two most important results of the S-L problem theory is the establishment that when the operator is in the correct form (to be shown below) that the eigenfunctions are mutually orthogonal with real eigenvalues. This obviously has immediate relevance for the Schrödinger equation.

Consider once again the linear differential equation

$$L(y) = f(x) \,, \tag{69}$$

the nonhomogeneous equation. We have already looked at the technique of the Variation of Parameters to solve for the particular solution $y_p(x)$ to this equation. We could also use the Laplace transform, which we haven't demonstrated here, and won't. But there is a subtle way to use the S-L theory here, using the method of the paraquant.

Suppose we have found the eigenfunctions ϕ_n and eigenvalues λ_n, such that

$$L(\phi_n) = \lambda_n r(x) \phi_n \,, \tag{70}$$

for the same operator L as in (69) and has a complete set of mutually orthogonal eigenfunctions. With this set we can write the usual piece-wise continuous functions we encounter, which show up such as $y(x)$ and $f(x)$, in terms of an infinite sum of these eigenfunctions. For example, set

$$y_p(x) = \sum_{n=1}^{\infty} c_n \phi_n \,. \tag{71}$$

Now, since for each n, $L(c_n\phi_n) = c_n L(\phi_n)$,

$$L(y_p) = L\Big(\sum_{n=1}^{\infty} c_n \phi_n\Big) = \sum_{n=1}^{\infty} c_n L(\phi_n) \,. \tag{72}$$

Now we replace the result of (72) into (69), and replace $L(\phi_n)$ by $\lambda_n r(x)\phi_n$ as in (70), to get:

$$\sum_{n=1}^{\infty} c_n \lambda_n r(x) \phi_n = f(x) \,. \tag{73}$$

This seemingly intractable equation will give up everything we want because of the convenience of orthogonality of the eigenfunctions. To see this, we multiply through by ϕ_m and integrate:

$$\sum_{n=1}^{\infty} c_n \lambda_n \int_a^b r(x)\phi_n \phi_m \, dx = \int_a^b f(x)\phi_m \, dx \,. \tag{74}$$

But, because of mutual orthogonality of the eigenfunctions, only one term of the sum on the LHS will survive, and that's when $n = m$, yielding

$$c_m \lambda_m \int_a^b r(x)\phi_m \phi_m \, dx = \int_a^b f(x)\phi_m \, dx \,. \tag{75}$$

If the eigenfunctions have been normalized, then $\int_a^b r(x)\phi_m\phi_m \, dx = 1$, but if not, we can still solve for c_m to get:

$$c_m = \frac{\int_a^b f(x)\phi_m \, dx}{\lambda_m \int_a^b r(x)\phi_m^2 \, dx} \,, \tag{76}$$

where we assume that none of the λ_m's is zero. Finally, for the particular solution, we have

$$y_p(x) = \sum_{m=1}^{\infty} \frac{\int_a^b f(x)\phi_m \, dx}{\lambda_m \int_a^b r(x)\phi_m^2 \, dx} \, \phi_m \,. \tag{77}$$

We solve a similar, but more challenging, problem in Appendix 4.

Part II: The standard form and converting to it

We take as our standard form of the S-L second-order linear differential equation the following

$$(p(x)y')' + q(x)y = \lambda r(x) y \,. \tag{78}$$

where, in the *regular* S-L problem, $p(x)$, $q(x)$ and $r(x)$ are greater than zero on our domain of interest, often to be a closed interval of the real line, such as $[a, b]$. Nearly every useful second-order linear differential equation can be put into this form. The exceptions of major interests are the *singular* equations, in which $p(x)$ may be zero on the boundary points.

Proof: Let's start with a general form for the differential equation

$$a(x)y'' + b(x)y' + c(x)y = \lambda d(x)y \,, \tag{79}$$

where $a(x)$ is greater than zero on the interval $[a, b]$. We ask how we can put (79) in S-L form. Step one: We now proceed with the standard use of an integrating factor $\mu(x)$ (also greater than zero on interval $[a, b]$) applied to (79):

$$\mu(x)a(x)y'' + \mu(x)b(x)y' + \mu(x)c(x)y = \mu(x)\lambda d(x)y \,, \tag{80}$$

Expanding (78), we get

$$p(x)y'' + p'(x)y' + q(x)y = \lambda r(x)y \,. \tag{81}$$

Comparing coefficients between (80) and (81), we get

$$p(x) = \mu(x)a(x) \,, \quad p'(x) = \mu(x)b(x) \,. \tag{82}$$

Combining these we get

$$\frac{p'(x)}{p(x)} = D_x \ln p(x) = \frac{b(x)}{a(x)} \,. \tag{83}$$

Integration gives $p(x)$:

$$p(x) = e^{\int [b(x)/a(x)]\, dx} \,. \tag{84}$$

To complete the conversion, we have

$$\mu(x) = \frac{p(x)}{a(x)} \,, \qquad q(x) = \frac{c(x)p(x)}{a(x)} \,, \qquad r(x) = \frac{d(x)p(x)}{a(x)} \,. \tag{85}$$

Part III: Self-adjoint operators, Lagrange Identity, Green's Identity
Now we begin our analysis on the linear operator version of the S-L equation

$$L(y) = (p(x)y')' + q(x)y(x) = \lambda r(x)y(x) \,. \tag{86}$$

where $L(\cdot) = (p(x)(\cdot)')' + q(x)(\cdot)$ is the Sturm-Liouville operator. Predictably, we convert (86) to spinor form, suppressing the understood dependence on the variable x:

$$L(\phi^\alpha) = (p\phi^{\alpha\prime})' + q\phi^\alpha = \lambda^{(\alpha)} r\phi^\alpha \,, \tag{87}$$

where, this time, α is a countable index.

Of course, complete solutions to differential equations require some boundary conditions. The S-L boundary conditions appear somewhat odd in their formal (homogeneous) presentation as

$$\alpha_1\phi(a) + \alpha_2\phi'(a) = 0\,, \tag{88a}$$

$$\beta_1\phi(b) + \beta_2\phi'(b) = 0\,, \tag{88b}$$

where not both α's can be zero and not both β's can be zero. These boundary conditions must apply to every eigenfunction, forcing the boundary terms ($[pW]_a^b$) of any two eigenfunctions to be zero! (This is worth spending some time to prove.) Why this is important will be made clear very soon.

There are three named types of boundary conditions. First, when $\alpha_2 = \beta_2 = 0$, the B.C. are referred to as *Dirichlet*, and $\phi(a) = \phi(b) = 0$. When $\alpha_1 = \beta_1 = 0$, the B.C. are referred to as *Neumann*, and $\phi'(a) = \phi'(b) = 0$. When $\phi(a) = \phi(b)$ and $\phi'(a) = \phi'(b)$ the B.C. are referred to as *periodic* (in which case, to force $[pW]_a^b$ to be zero, we must have that $p(b) = p(a)$). In the case where neither α_1 nor α_2 is zero, the proof that the Wronskian is zero at point a is provided in Appendix 5.

At this point, the literature on S-L theory divides into two highly divergent paths: The High Road and the Low Road. On the High Road, the theory of abstract self-adjoint operators is introduced and certain results proved there. Then the S-L operator is proven to be self-adjoint using

$$L(\phi^\alpha) = (p\phi^{\alpha\prime})' + q\phi^\alpha\,, \tag{89}$$

which comes from (87). The Low Road ignores all mention of self-adjoint operators and just proves everything by appealing to this equation

$$(p\phi^{\alpha\prime})' + q\phi^\alpha = \lambda^{(\alpha)} r\phi^\alpha\,, \tag{90}$$

which also comes from (87).

Let's multiply (89) through by ϕ_α and sum, getting

$$\phi_\alpha L(\phi^\alpha) = \phi_\alpha(p\phi^{\alpha\prime})'\,, \tag{91}$$

where $\alpha = m, n$, and the components of ϕ^α are any two eigenfunctions of L, namely, ϕ_m and ϕ_n. Now, contrafluxing the RHS gives

$$(\phi_\alpha p\phi^{\alpha\prime})' - \phi_\alpha{}' p\phi^{\alpha\prime}\,, \tag{92}$$

where the contra term is zero because of the isotropic nature of $\phi^{\alpha\prime}$, leaving us with

$$\phi_\alpha L(\phi^\alpha) = (\phi_\alpha p\phi^{\alpha\prime})' = (-pW)'\,, \tag{93}$$

the so-called *Lagrange Identity*. On integrating this, we get the so-called *Green's Identity*

$$\int_a^b \phi_\alpha L(\phi^\alpha)\,dx = -(pW)|_a^b\,, \tag{94}$$

or

$$\int_a^b \phi^\alpha L(\phi_\alpha)\, dx = (pW)|_a^b \,. \tag{95}$$

Corollary: When the S-L B.C. are applied to this, $(pW)|_a^b = 0$, yielding

$$\int_a^b \phi^\alpha L(\phi_\alpha)\, dx = 0 \,. \tag{96}$$

which proves that L is self-adjoint. That is, (96) is equivalent to the claim

$$\langle u, L(v)\rangle = \langle L(u), v\rangle \,. \tag{97}$$

To compare our spinor derivation of the Lagrange Identity to the conventional one, see Appendix 6.

Notation
When an overbar or overline occurs on some object, it means to take the complex conjugate of it.

Part IV: mutually orthogonal eigenfunctions, real eigenvalues
Continuing on the Low Road, if we multiply (90) through by ϕ_α and sum we get:

$$(pW)' = \lambda^{(\alpha)} r \phi_\alpha \phi^\alpha \,. \tag{98}$$

Now, with $\phi^1 \mapsto \phi_m$ and $\phi^2 \mapsto \phi_n$, the RHS of (98) expands to

$$(\lambda_n - \lambda_m) r(x) \phi_n \phi_m \,, \tag{99}$$

where m and n represent any two eigen-numbers, not indices to be summed on. Substituting this into (98) and integrating from a to b and using the S-L B.C., yields

$$(\lambda_n - \lambda_m) \int_a^b r(x) \phi_n \phi_m dx = 0 \,. \tag{100}$$

If these two eigenvalues are distinct, then

$$\int_a^b r(x) \phi_n \phi_m dx = 0 \,, \tag{101}$$

and thus the eigenfunctions ϕ_n and ϕ_m, corresponding to distinct eigenvalues, are formally 'orthogonal' on the chosen interval with weight function $r(x)$.

Now for our second major result: Referring back to (100), set $\phi_m \mapsto \overline{\phi_n}$ and, $\lambda^{(m)} \mapsto \overline{\lambda_n}$. Then

$$(\lambda_n - \overline{\lambda_n}) \int_a^b r(x) \phi_n \overline{\phi_n} dx = (\lambda_n - \overline{\lambda_n}) \int_a^b r(x) ||\phi_n||^2 dx = 0 \,. \tag{102}$$

Now, since $r(x) > 0$ and $||\phi_n||^2 \geq 0$ on $[a, b]$, then $\int_a^b r(x)||\phi_n||^2 dx > 0$, implying that $(\lambda_n - \overline{\lambda_n}) = 0$. And this implies that $\lambda_n = \overline{\lambda_n}$, and this implies that λ_n is real-valued.

Hence, the second reason about (constructing) self-adjoint differential operators is that their eigenvalues are real. In the case of quantum mechanics, where eigenvalues correspond to physical measurements, this is an obvious good thing.

8 On the Sturm-Liouville literature.

I take no pleasure to confess that my experience learning the Sturm-Liouville theory from the various sources available to me in books and papers on line has not been a bucolic journey. There seems to be a lot of confusion about what constitutes the essentials of the subject, and strong disagreement on how to present even the fundamentals of the theory. For example, some authors make a big deal about self-adjoint operators, derive important results using abstract self-adjoint operators, and then prove that the S-L operator is self-adjoint. Other authors never even mention self-adjoint operators, presenting the same result on the Low Road. Some authors equate the Hermitian and self-adjoint operators; others make a distinction. Authors disagree on what should be called the Lagrange Identity and Green's Identity. When dealing with self-adjoint operators, some authors include complex conjugation in the inner products, others ignore it, but with no explanation why. At the time of this writing, I am still trying to sort it all out, so that I may have to amend my own presentation of the S-L theory when it is clearer in my own mind.

9 Green's Function Approach to the Nonhomogeneous Equation

We begin with the equation that we desire to solve for the particular solution y_p to the nonhomogeneous equation:

$$L(y_p) = (py_p')' + qy_p = j(x)\,, \tag{103}$$

which expands to

$$L(y_p) = py_p{}'' + p'y_p' + qy_p = j(x)\,. \tag{104}$$

We wish to construct a function $G(x, \xi)$, called a *Green's Function*, that satisfies the equations

$$y_p = \int_a^b G(x, \xi) j(\xi)\, d\xi \quad \text{where} \quad \xi \in (a, b) \tag{105}$$

and paraquant

$$L(G(x, \xi)) = \delta(x - \xi)\,. \tag{106}$$

Once $G(x,\xi)$ is found (as the solution to (106)), we can determine (105). Why does this work? Because, applying $L(\cdot)$ across (105), we get

$$L(y_p) = \int_a^b L(G(x,\xi))j(\xi)d\xi = \int_a^b \delta(x-\xi)j(\xi)d\xi = j(x)\,. \tag{107}$$

Corresponding to the homogeneous form of (103), we have

$$L(\phi^\alpha) = (p\phi^{\alpha\prime})' + q\phi^\alpha = 0\,, \tag{108}$$

for any two of its homogeneous solutions ϕ^α $(\alpha = 1, 2)$.

$G(x,\xi)$ must conform to certain features of the homogenous problem. It must be continuous in x and have first and second partials (G_x and G_{xx}) continuous on x on the interval (a,b), except at some special interior point $x = \xi$. And, except for the point $x = \xi$, $L(G) = 0$, leading us to suspect that we can construct an ansatz for G out of the homogeneous solutions.

We can build G out of ϕ^1 and ϕ^2 by using (108). We need $L(G)$ to be different from zero at $x = \xi$, but zero elsewhere on the interval. Thus we can use ϕ^1 to construct G for $x \le \xi$ and use ϕ^2 for $x \ge \xi$.

Now, the notion of the delta function is only made precise inside an integral. So, using L from (104) applied to (106) and integrating, we have

$$\int_{\xi-\epsilon}^{\xi+\epsilon} [\,p(x)G_{xx} + p'(x)G_x + qG\,]\,dx = \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)dx = 1\,, \tag{109}$$

where ϵ is a small positive real number that we will in time take to zero. The spike function comes from the G_{xx} term, the only term in the integrand to survive when ϵ goes to zero. So,

$$\int_{\xi-\epsilon}^{\xi+\epsilon} p(x)G_{xx}dx = 1\,. \tag{110}$$

From which we get for small ϵ, $p(x) \approx p(\xi)$, therefore

$$p(\xi)[G_x(\xi+\epsilon) - G_x(\xi-\epsilon)] = 1\,. \tag{111}$$

Thus we see that G_x, the slope of G, suffers a discontinuity at the point ξ. Then,

$$G_x(\xi+\epsilon) - G_x(\xi-\epsilon)] = \frac{1}{p(\xi)}\,. \tag{112}$$

We try the ansatz using the solutions to the homogeneous equation (108)

$$G(x,\xi) = \begin{cases} c_1\phi^1(x), & x \le \xi \\ c_2\phi^2(x), & x \ge \xi \end{cases} \tag{113}$$

where c_1, c_2 are independent of x, but may depend on ξ, and $\phi^1(x)$ satisfies a homogeneous boundary condition at a, and $\phi^2(x)$ satisfies a homogeneous boundary condition at b.

At $x = \xi$, by continuity of $G(x, \xi)$, $c_1\phi^1(\xi) = c_2\phi^2(\xi)$ therefore

$$\mu_\alpha \phi^\alpha(\xi) = 0\,, \tag{114}$$

where $\mu_\alpha \equiv (c_1, -c_2)$. (And, yes, I have intensionally constructed an isotrope out of thin air. Let's see if it helps.) Now, $G(x, \xi)$ in (113) must also satisfy (112):

$$c_2\phi^{2\prime}(\xi + \epsilon) - c_1\phi^{1\prime}(\xi - \epsilon) = \frac{1}{p(\xi)} \tag{115}$$

or, with $\epsilon \to 0$,

$$\mu_\alpha \phi^{\alpha\prime}(\xi) = \frac{-1}{p(\xi)}\,. \tag{116}$$

So, we solve Eqs. (114) and (116) together, which is similar to the situation we had to solve for coupled equations in the Variation of Parameters problem.

Again we start with the isotrope (114), setting

$$\mu_\alpha(\xi) = \kappa(\xi)\phi_\alpha(\xi) \tag{117}$$

and solve for $\kappa(\xi)$. First, we multiply (117) through by $\phi^{\alpha\prime}(\xi)$ and solve for $\kappa(\xi)$ to get

$$\kappa(\xi) = \frac{1}{p(\xi)W(\xi)}\,, \tag{118}$$

where $W(\xi) = -\phi_\alpha(\xi)\phi^{\alpha\prime}(\xi)$. Notice in the denominator of (118) that we again have the now familiar pW that we had in the Variation of Parameters problem.

So then (117) becomes

$$\mu_\alpha(\xi) = \frac{\phi_\alpha(\xi)}{p(\xi)W(\xi)}\,. \tag{119}$$

Thus (113) becomes

$$G(x, \xi) = \begin{cases} \dfrac{\phi_1(\xi)\phi^1(x)}{p(\xi)W(\xi)}, & x \le \xi \\ -\dfrac{\phi_2(\xi)\phi^2(x)}{p(\xi)W(\xi)}, & x \ge \xi \end{cases} \tag{120}$$

Or, on raising the spinor indicies,

$$G(x, \xi) = \begin{cases} \dfrac{\phi^2(\xi)\phi^1(x)}{p(\xi)W(\xi)}, & x \le \xi \\ \dfrac{\phi^1(\xi)\phi^2(x)}{p(\xi)W(\xi)}, & x \ge \xi \end{cases} \tag{121}$$

Corollary: $G(x, \xi) = G(\xi, x)$.
Proof: First, remember that $pW =$ constant, therefore, when we interchange x and ξ in (121), being mindful that the directions of the inequalities also reverse, we get that $G(x, \xi) = G(\xi, x)$.

10 Green Function via the Variation of Parameters.

This section is inspired by the online paper in [6] ("Boundary value problems and Green's functions"). My presentation here is with isotropic spinors, and whether or not this has increased the comprehensibility of the subject, I must leave to the reader to decide. Anyway, it is a natural extension of the sections above on the Variation of Parameters and self-adjoint operators already discussed.

Given the second-order differential equation (49), presented again (changing the inhomogeneous term to $f(x)$)

$$y'' + a_1(x)y' + a_2(x)y = f(x)\,, \tag{122}$$

with particular solution (62), presented again here (with the addition of a lower limit of integration),

$$y_p(x) = -\int_a^x \frac{f(\tau)\phi_\alpha(\tau)\phi^\alpha(x)}{W(\tau)}\,d\tau\,, \tag{123}$$

we seek to construct a Green function to the general solution

$$y(x) = c_1\phi^1 + c_2\phi^2 + y_p \tag{124}$$

out of the homogeneous solutions $\phi^1(= y_1)$ and $\phi^2(= y_2)$ and particular solution y_p from (123).

The difference between the earlier treatment and this one is that we are adding standard 2-point homogeneous boundary conditions (B.C.'s). The **big** question we ask is this: Is it possible, given appropriate boundary conditions, to express the solution for $y(x)$ in (124) as, say, a single definite integral from a to b, and thus avoid solving for c_1 and c_2 on a problem-by-problem basis? The answer is *yes*, and the Green function we get is merely a means to that end. So let's prove this.

The homogeneous solutions are required to satisfy only one boundary condition at a time, so that $B_a[\phi^1] = 0$ and $B_b[\phi^1] \neq 0$, and, of course, $B_a[\phi^2] \neq 0$ and $B_b[\phi^2] = 0$, where for parameters α_1, α_2, for boundary point ξ, and for generic function ϕ,

$$B_\xi[\phi] = (\alpha_1(\xi), \alpha_2(\xi))\begin{bmatrix}\phi(\xi)\\ \phi'(\xi)\end{bmatrix} = \alpha_1(\xi)\phi(\xi) + \alpha_2(\xi)\phi'(\xi)\,. \tag{125}$$

To construct a Green function for $y(x)$ we need $y(x)$ to satisfy both homogeneous boundary conditions

$$B_a[y] = 0 \quad \text{and} \quad B_b[y] = 0\,. \tag{126}$$

Fortunately, the boundary operator $B_\xi[\cdot]$ is linear and will distribute over the terms in (124).

In preparation for applying the constraints in (126), let's first investigate how our y_p fares under our two boundary constraints. By inspection of (123), it's clear that $y_p(a) = 0$, because of the upper limit of integration is a, the same as the lower limit. For our boundary operator, we also need the derivative of y_p, by differentiating (123):

$$y_p'(x) = -\left[\frac{f(x)\phi_\alpha(x)\phi^\alpha(x)}{W(x)} + \int_a^x \frac{b(\tau)\phi_\alpha(\tau)\phi^{\alpha\prime}(x)}{W(\tau)}\, d\tau\right], \tag{127}$$

where we have used the product rule and also have taken into account the fact that we are differentiating an integral with a variable limit of integration. However, the first term on the RHS of (127) is zero because ϕ^α is isotropic. Therefore,

$$y_p'(x) = -\int_a^x \frac{f(\tau)\phi_\alpha(\tau)\phi^{\alpha\prime}(x)}{W(\tau)}\, d\tau\,. \tag{128}$$

So, clearly $y_p'(a) = 0$ and that together with $y_p(a) = 0$ (which we just determined) implies that $B_a[y_p] = 0$.

Now we investigate what happens at point b.

$$\begin{aligned}
-B_b[y_p] &= \alpha_1(b)\int_a^b \frac{f(\tau)\phi_\alpha(\tau)\phi^\alpha(b)}{W(\tau)}\, d\tau + \alpha_2(b)\int_a^b \frac{f(\tau)\phi_\alpha(\tau)\phi^{\alpha\prime}(b)}{W(\tau)}\, d\tau \\
&= \int_a^b \frac{f(\tau)\phi_\alpha(\tau)\begin{bmatrix}\alpha_1(b)\phi^1\\ \alpha_1(b)\phi^2\end{bmatrix}}{W(\tau)}\, d\tau + \int_a^b \frac{f(\tau)\phi_\alpha(\tau)\begin{bmatrix}\alpha_2(b)\phi^{1\prime}\\ \alpha_2(b)\phi^{2\prime}\end{bmatrix}}{W(\tau)}\, d\tau \\
&= \int_a^b \frac{f(\tau)\phi_\alpha(\tau)\begin{bmatrix}B_b[\phi^1]\\ 0\end{bmatrix}}{W(\tau)}\, d\tau \qquad\qquad (\text{since } B_b[\phi^2] = 0) \\
&= \int_a^b \frac{f(\tau)\phi_1(\tau)B_b[\phi^1]}{W(\tau)}\, d\tau \\
&= B_b[\phi^1]\int_a^b \frac{f(\tau)\phi_1(\tau)}{W(\tau)}\, d\tau\,.
\end{aligned} \tag{129}$$

We now have enough information to solve for the coefficients of $y(x)$ in (124). Applying the constraint $B_a[y] = 0$, we have that

$$B_a[y] = c_1 B_a[\phi^1] + c_2 B_a[\phi^2] + B_a[y_p] = c_1 \cdot 0 + c_2 B_a[\phi^2] + 0 = 0\,, \tag{130}$$

from which we conclude that $c_2 = 0$. That leaves us with constraint $B_b[y] = 0$ to satisfy.

$$B_b[y] = c_1 B_b[\phi^1] + B_b[y_p] = 0\,. \tag{131}$$

Using (129), we can finish our determination of the coefficients by setting

$$c_1 = \int_a^b \frac{f(\tau)\phi_1(\tau)}{W(\tau)}\, d\tau\,. \tag{132}$$

Applying what we have found into (124), gives us

$$
\begin{aligned}
y(x) &= \int_a^b \frac{f(\tau)\phi_1(\tau)\phi^1(x)}{W(\tau)}\,d\tau - \int_a^x \frac{f(\tau)\phi_\alpha(\tau)\phi^\alpha(x)}{W(\tau)}\,d\tau \\
&= \int_a^x \frac{f(\tau)\phi_1(\tau)\phi^1(x)}{W(\tau)}\,d\tau + \int_x^b \frac{f(\tau)\phi_1(\tau)\phi^1(x)}{W(\tau)}\,d\tau \\
&\quad - \int_a^x \frac{f(\tau)\phi_\alpha(\tau)\phi^\alpha(x)}{W(\tau)}\,d\tau \qquad \text{(we broke up the 1st integral of the previous line and we'll simplify by combining the 1st and 3rd terms)} \\
&= -\int_a^x \frac{f(\tau)\phi_2(\tau)\phi^2(x)}{W(\tau)}\,d\tau + \int_x^b \frac{f(\tau)\phi_1(\tau)\phi^1(x)}{W(\tau)}\,d\tau \qquad \text{(now, we raise the lower indices)} \\
&= \int_a^x \frac{f(\tau)\phi^1(\tau)\phi^2(x)}{W(\tau)}\,d\tau + \int_x^b \frac{f(\tau)\phi^2(\tau)\phi^1(x)}{W(\tau)}\,d\tau\,. \qquad (133)
\end{aligned}
$$

Thus we can write the solution of (124), with B.C.'s applied, as a definite integral, using the Green function $G(x,\tau)$, in the form

$$
y(x) = \int_a^b f(\tau)G(x,\tau)\,d\tau\,, \qquad (134)
$$

where

$$
G(x,\tau) = \begin{cases} \dfrac{\phi^1(\tau)\phi^2(x)}{W(\tau)} & \text{for} \quad a \le \tau \le x \le b, \\[2ex] \dfrac{\phi^1(x)\phi^2(\tau)}{W(\tau)} & \text{for} \quad a \le x \le \tau \le b. \end{cases} \qquad (135)
$$

The amazing part of this solution is that the boundary conditions have been so cleverly concealed within the solution that they only seem to hint at their existence at all through the limits of integration of the integral in (134).

11 Conclusion

The isotropic nature of the spinors we defined has afforded us a significant reduction in the algebraic complexity of our calculations to find solutions to our problems. The symplectic inner product allows us to remove terms by multiplication (with summation) and to hide bulky determinants inside every symplectic inner product. The Main Heuristic teaches us how to split isotropes, allowing us to solve for one of the spinor factors in terms of the other. Add to these space-saving techniques the summation convention, and we have the basis for some compact and beautiful proofs.

Ironically, however, though the final spinor forms of the results I obtained are compact and beautiful in themselves, I often have to expend much effort and space to prove that my correct spinor results are equivalent to the conventional results. However, a computer would have no problem using the spinor formulas as soon as it's been taught the spinor rules.

12 Appendix: Spinor Cheat Sheet

Given spinor $A^\alpha \mapsto \begin{bmatrix} A^1 \\ A^2 \end{bmatrix}$, then

$$A_\alpha \mapsto [A_1, A_2] = [A^2, -A^1]\,. \tag{136}$$

Given spinor $B_\alpha \mapsto [B_1, B_2]$, then

$$B^\alpha \mapsto \begin{bmatrix} -\,B_2 \\ B_1 \end{bmatrix}\,. \tag{137}$$

Given spinor $\widehat{A}^\alpha \mapsto \begin{bmatrix} \bar{A}^1 \\ A^2 \end{bmatrix}$, then

$$\widehat{A}_\alpha \mapsto [A_1, \bar{A}_2]\,. \tag{138}$$

13 Appendix: Relation to Simple Algebra, Geometric Algebra

For the sake of satisfying my curiosity, I searched for a way to use isotropic spinors to solve for a system of two linear equations in two unknowns. I found three different methods so far.

Say we want to solve for ϕ^1 and ϕ^2 (instead of x and y, of course) in the following general system of equations

$$v_1\phi^1 + v_2\phi^2 = v_\alpha\phi^\alpha = a \tag{139a}$$

$$u_1\phi^1 + u_2\phi^2 = u_\alpha\phi^\alpha = b \tag{139b}$$

where we assume that the coefficients and the constants a and b are given and different from zero generally, and $v_\alpha u^\alpha \neq 0$. Of course, we could just set up the matrix equation and either use Cramer's Rule or just take a matrix inverse. But the spinor solution is a more interesting way to invert the equations.

Method One: The Most Elegant
I'll begin with the method I think is elegant, which begins, predictably, by constructing an isotrope. Divide (139a) by a and (139b) by b to get

$$\widetilde{v}_1\phi^1 + \widetilde{v}_2\phi^2 = \widetilde{v}_\alpha\phi^\alpha = 1 \tag{140a}$$

$$\widetilde{u}_1\phi^1 + \widetilde{u}_2\phi^2 = \widetilde{u}_\alpha\phi^\alpha = 1 \tag{140b}$$

where $\widetilde{v}_\alpha = v_\alpha/a$ and $\widetilde{u}_\alpha = u_\alpha/b$. Now, just subtract (140b) from (140a) to get

$$w_\alpha\phi^\alpha = 0 \tag{141}$$

where $w_\alpha = \widetilde{v}_\alpha - \widetilde{u}_\alpha$. Therefore, by the isotropy of the spinors,

$$\phi^\alpha = \lambda w^\alpha\,, \tag{142}$$

where λ is a scaling factor we can easily solve for: First, multiply (142) through by either v_α or u_α, using (139a) or (139b), respectively, and sum, and then substitute that result for λ back into (142). Let's multiply through by v_α, and substitute for the λ expression into (142) to get the result

$$\phi^\alpha = \frac{aw^\alpha}{v_\alpha w^\alpha} = \frac{au^\alpha - bv^\alpha}{v_\alpha u^\alpha}\,. \tag{143}$$

and that's the complete and correct answer, though to prove that it is equivalent to using Cramer's rule instead on (139a) and (139b) requires a little arithmetic. It's actually much easier to prove that they satisfy the original equations (139a) and (139b).

Method Two: The Messiest

Since everything seems to depend on getting an inner product of spinors equal to zero, let's setup a system of equations that already has one!

$$v_\alpha \phi'^\alpha = a' \tag{144a}$$

$$u_\alpha \phi'^\alpha = 0 \tag{144b}$$

(Primes are not derivatives.) As before, from (144b), we know that $\phi'^\alpha = \lambda u^\alpha$ and we are to solve for λ. So, we multiply through by v_α and sum, to get $\lambda = a'/v_\alpha u^\alpha$. Therefore,

$$\phi'^\alpha = \frac{a'}{v_\alpha u^\alpha} u^\alpha\,. \tag{145}$$

It's actually a simple matter to convert (139a) and (139b) to (144a) and (144b), respectively, by using a simple change of variables, given by

$$\phi^1 = \phi'^1 + b/u_1 \tag{146a}$$

$$\phi^2 = \phi'^2 \tag{146b}$$

The result of this transformation is to produce equations (144a) and (144b), with $a' = a - bv_1/u_1$. This gives the solution to our original equations (139a) and (139b) as

$$\begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix} = \begin{bmatrix}\phi'^1\\ \phi'^2\end{bmatrix} + \begin{bmatrix}b/u_1\\ 0\end{bmatrix} = \frac{a'}{v_\alpha u^\alpha}\begin{bmatrix}u^1\\ u^2\end{bmatrix} + \begin{bmatrix}b/u_1\\ 0\end{bmatrix} \tag{147}$$

or

$$\begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix} = \begin{bmatrix}\kappa u_2 + b/u_1\\ -\kappa u_1\end{bmatrix} \tag{148}$$

where $\kappa = \dfrac{a'}{v_\alpha u^\alpha} = \dfrac{a - bv_1/u_1}{v_\alpha u^\alpha}$ and $u^1 \mapsto u_2$ and $u^2 \mapsto -u_1$. With considerable arithmetic, one can put (148) in standard form.

Method Three: The Shortest

This method is suggested by equation (143) that we try an ansatz of the form

$$\phi^\alpha = Au^\alpha + Bv^\alpha\,. \tag{149}$$

Then A and B are easily determined separately by multiplying through by u_α and v_α in turn (and summing, of course), using the isotropy of the spinors, to get, after combining terms, as before

$$\phi^\alpha = \frac{au^\alpha - bv^\alpha}{v_\beta u^\beta}\,, \tag{150}$$

where the β's are summed on. In other words, generically expressed

$$\sum_\alpha C_\alpha D^\alpha = \sum_\beta C_\beta D^\beta\,, \tag{151}$$

or dropping the summation symbols

$$C_\alpha D^\alpha = C_\beta D^\beta\,. \tag{152}$$

So, when an index is summed on, it really doesn't matter what variable is used for the summation index, so long as no confusion arises.

Relation to Linear and Geometric Algebra
OK, the proof is impressively short, but couldn't we do something like that by interpreting (149) as an equation on Euclidean vectors? Let's try it. We convert the spinor ansatz to the vector ansatz

$$\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}\,, \tag{153}$$

with 'constraints' $\mathbf{v}\cdot\mathbf{x} = a$ and $\mathbf{u}\cdot\mathbf{x} = b$, and no prior constraint on the relationship of $\mathbf{u}$ to $\mathbf{v}$. So, let's apply the constraints on c_1 and c_2:

$$\mathbf{v}\cdot\mathbf{x} = a = c_1\mathbf{v}\cdot\mathbf{u} + c_2\mathbf{v}^2\,, \tag{154a}$$

$$\mathbf{u}\cdot\mathbf{x} = b = c_1\mathbf{u}^2 + c_2\mathbf{u}\cdot\mathbf{v}\,, \tag{154b}$$

which is just another couple of equations in two unknowns to be solved for. So this approach hasn't helped us at all, unless $\mathbf{v}\cdot\mathbf{u} = 0$.

Although linear algebra was no help in solving (153), geometric algebra can do it (see [5]). We are given $\mathbf{v}\cdot\mathbf{x} = a$ and $\mathbf{u}\cdot\mathbf{x} = b$ then

$$\mathbf{x}\cdot\mathbf{u}\wedge\mathbf{v} = \mathbf{x}\cdot\mathbf{u}\,\mathbf{v} - \mathbf{x}\cdot\mathbf{v}\,\mathbf{u} = b\mathbf{v} - a\mathbf{u}\,. \tag{155}$$

But $\mathbf{x}\cdot\mathbf{u}\wedge\mathbf{v} = \mathbf{x}\,\mathbf{u}\wedge\mathbf{v}$. Therefore

$$\mathbf{x} = [b\mathbf{v} - a\mathbf{u}](\mathbf{u}\wedge\mathbf{v})^{-1}\,. \tag{156}$$

14 Appendix: Reduction of Order Technique

Problem 3: For the differential equation given in (34), if one solution is known, say $\phi^1(x)$, find a formula for the other solution. In the theory of ODEs this is referred to as a *reduction of order technique.*

Using the formal definition of the Wronskian and the value obtained for it in (37), we can write

$$W(x) = \phi^1 \phi^{2\prime} - \phi^2 \phi^{1\prime} . \tag{157}$$

Dividing this through by $(\phi^1)^2$ gives

$$\frac{W(x)}{(\phi^1)^2} = \frac{\phi^{2\prime}}{\phi^1} - \frac{\phi^{1\prime}\phi^2}{(\phi^1)^2} = \frac{d}{dx}\left(\frac{\phi^2}{\phi^1}\right) . \tag{158}$$

Integrating both sides and employing a bit of algebra gives us

$$\phi^2(x) = \left(\int^x \frac{W(x)}{[\phi^1(x)]^2}\, dx + c\right) \phi^1(x) , \tag{159}$$

where c is an arbitrary constant. Of course we're assuming that $\phi^1(x)$ is not zero on our domain of interest.

The curious reader may have noticed that, unlike $a_1(x)$, which appears explicitly in this solution via $W(x)$ in (37), $a_2(x)$ does *not* explicitly appear. This solution for $\phi^2(x)$ has the false appearance that $a_2(x)$ didn't matter at all. But that's not the case. The 'hard work' in getting both solutions to (34) by this approach was in getting the first solution $\phi^1(x)$, and we would have had to use $a_2(x)$ to get that.

The formula in (159) is often derived from a slightly different starting point. The form of the solution in (159) suggests that we try instead the ansatz

$$\phi^2 = u(x)\phi^1 , \tag{160}$$

from which we can calculate the Wronskian as

$$W = \begin{vmatrix} \phi^1 & u\phi^1 \\ \phi^{1\prime} & u'\phi^1 + u\phi^{1\prime} \end{vmatrix} = u'(\phi^1)^2 . \tag{161}$$

After solving for u' and integrating to get $u(x)$ and then plugging that result into (160), we again get (159).

15 Appendix: More Difficult Nonhomogeneous Problem

Here we add another term to the differential equation back in (69):

$$L(y) = \mu r(x) y + f(x) , \tag{162}$$

where μ is a fixed number. We again seek the particular solution $y_p(x)$ to this equation by finding the eigenfunctions and eigenvalues to this equation

$$L(\phi_n) = \lambda_n r(x) \phi_n , \tag{163}$$

for the same operator L as in (162).

$$y_p(x) = \sum_{n=1}^{\infty} c_n \phi_n \,. \tag{164}$$

Proceeding as before, this time we get

$$\sum_{n=1}^{\infty} c_n \lambda_n r(x) \phi_n = \sum_{n=1}^{\infty} c_n \mu r(x) \phi_n + f(x) \,. \tag{165}$$

Multiplying through by ϕ_m and integrating gives:

$$c_m(\lambda_m - \mu) \int_a^b r(x) \phi_m \phi_m \, dx = \int_a^b f(x) \phi_m \, dx \quad \text{(no sum on } m) \,. \tag{166}$$

So, if none of the eigenvalues c_m is equal to μ, then:

$$c_m = \frac{\int_a^b f(x) \phi_m \, dx}{(\lambda_m - \mu) \int_a^b r(x) \phi_m^2 \, dx} \,, \tag{167}$$

for all m. Finally, for the particular solution, we have

$$y_p(x) = \sum_{m=1}^{\infty} \frac{\int_a^b f(x') \phi_m(x') \, dx'}{(\lambda_m - \mu) \int_a^b r(x') \phi_m^2(x') \, dx'} \, \phi_m(x) \,. \tag{168}$$

16 Appendix: Proof of Wronskian Being Zero

Consider the case when neither α_1 nor α_2 is zero in the boundary conditions in (88a) at point a.

$$W = \begin{vmatrix} \phi^1 & \phi^2 \\ \phi^{1\prime} & \phi^{2\prime} \end{vmatrix} \,, \tag{169}$$

where ϕ^1 and ϕ^2 are any two eigenfunctions of the linear operator L. For all eigenfunctions at point a we can write

$$\phi^{1\prime}(a) = -\frac{\alpha_1}{\alpha_2} \, \phi^1(a) \,, \qquad \phi^{2\prime}(a) = -\frac{\alpha_1}{\alpha_2} \, \phi^2(a) \,. \tag{170}$$

Substituting these into (169), yeilds

$$W = \begin{vmatrix} \phi^1(a) & \phi^2(a) \\ -\dfrac{\alpha_1}{\alpha_2} \, \phi^1(a) & -\dfrac{\alpha_1}{\alpha_2} \, \phi^2(a) \end{vmatrix} \,. \tag{171}$$

The proof that W in this last equation is zero follows either by direct calculation, or, as proved in linear algebra, because the rows of the determinant are not linearly independent.

17 Appendix: Conventional Derivation of Lagrange Identity

Prove the Lagrange Identity

$$uL(v) - vL(u) = [p(uv' - vu')]' = (pW)' \,, \tag{172}$$

where u and v are any two eigenfunctions of $L(y) = (py')' + qy$ and $W = uv' - vu'$:

$$\begin{aligned} uL(v) - vL(u) &= u[(pv')' + qv] - v[(pu')' + qu] \\ &= u(pv')' - v[(pu')' \\ &= [(upv')' - u'pv'] - [(vpu')' - v'pu'] \\ &= (upv')' - (vpu')' \\ &= [p(uv' - vu')]' \\ &= (pW)' \end{aligned}$$

For comparison, the spinor proof goes like this again, with $L(\phi^\alpha) = (p\phi^{\alpha\prime})' + q\phi^\alpha$ and $W = \phi^\alpha {\phi_\alpha}' = -\phi_\alpha \phi^{\alpha\prime}$:

$$\phi_\alpha L(\phi^\alpha) = \phi_\alpha (p\phi^{\alpha\prime})' = (\phi_\alpha p \phi^{\alpha\prime})' = (-pW)' \,, \tag{173}$$

or

$$\phi^\alpha L(\phi_\alpha) = (pW)' \,, \tag{174}$$

18 Appendix: Theorem on Uniqueness of Eigenfunction per Eigenvalue

Theorem: In regular S-L problems where the Wronskian is zero on the boundary points, the eigenfunctions corresponding to a given eigenvalue are unique up to a scalar multiplicative factor.

Proof: Assume that ϕ^1 and ϕ^2 have the same eigenvalue $\lambda^{(1)} = \lambda^{(2)} = \lambda$. Then, multiplying (90) through by ϕ_α and summing, gives

$$(pW)' = \lambda^{(\alpha)} r \phi_\alpha \phi^\alpha = \lambda r \phi_\alpha \phi^\alpha = 0 \,, \tag{175}$$

since ϕ^α is isotropic. But from calculus, we can write that $p(x)W(x) = \text{const}$ on the entire interval $[a, b]$. However, because of the boundary conditions, which all eigenfunctions must obey, $W(a) = 0$ implies that $p(x)W(x) = 0$ on $[a, b]$, which requires that $W(x) = 0$ on $[a, b]$. By Lemma 1e, we conclude that ϕ^2 is a scalar multiple of ϕ^1.

19 Appendix: Theorem on solutions to 2nd-order linear differential equation with constant coefficients, double roots

Theorem: Let

$$ay'' + by' + cy = 0 \tag{176}$$

be a 2nd-order linear differential equation with constant coefficients. Then, if we let y_1 be one solution, another solution is xy_1.

Proof:

From theory of ODEs, we know that (176) has two linearly independent solutions y_1 and y_2, say. We can find these solutions, in most cases, by using the ansatz $y = ke^{rx}$, where r and k are complex numbers. Plugging this trial solution into (176) gives us, after some canceling,

$$ar^2 + br + c = 0\,. \tag{177}$$

From the quadratic equation we get

$$r\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{178}$$

We have three cases:
Case 1: If $b^2 - 4ac > 0$, we get two distinct real roots, and the general solution to (176) is

$$y(x) = c_+ e^{r_+ x} + c_- e^{r_- x}\,, \tag{179}$$

where c_+ and c_- are arbitrary complex numbers.

Case 2: If $b^2 - 4ac < 0$, we get two complex conjugate roots, and the general solution to (176) is

$$y(x) = c_+ e^{\mu x} e^{i\gamma x} + c_- e^{\mu x} e^{-i\gamma x}\,, \tag{180}$$

where c_+ and c_- are arbitrary complex numbers, $\mu \equiv -b/2a$, and $\gamma = \frac{\sqrt{|b^2-4ac|}}{2a}$.

Case 3: If $b^2 - 4ac = 0$, we get a double root, and the general solution to (176) is

$$y(x) = c_1 e^{\mu x} + c_2 x e^{\mu x}\,, \tag{181}$$

where c_1 and c_2 are arbitrary complex numbers.

It's this last case I will take the time to prove. Obviously,

$$y_1 = Ce^{rx} \tag{182}$$

is a solution, where $r = -b/2a$. We're going to use a technique similar to Variation of Parameters to solve for y_2. So, we try the ansatz $y_2 = v(x)y_1$ into (176) to see what differential equation v must satisfiy. When we do, we get

$$v[ay_1'' + by_1' + cy_1] + av''y_1 + (by_1 + 2ay_1')v' = 0\,. \tag{183}$$

Now, the expression inside the square brackets is zero because of (176). The expression inside the parentheses is zero because it satisfies the relation obtained by differentiating (182). Thus (183) boils down to

$$av''y_1 = 0\,. \tag{184}$$

But since neither a nor y_1 is identically zero, then $v'' = 0$, which has solution

$$v(x) = C_1 x + C_2\,. \tag{185}$$

This leaves us with the solution for y_2

$$y_2(x) = (C_1 x + C_2) y_1(x) = (C_1 x + C_2) e^{rx}\,. \tag{186}$$

Therefore the general solution to (176) for the case of a double root is

$$y(x) = C_0 e^{rx} + (C_1 x + C_2) e^{rx}\,, \tag{187}$$

which simplifies to

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}\,. \tag{188}$$

References

[1] E.A. Coddington, *An Introduction to Ordinary Differential Equations*, Prentice-Hall (1961), 111–124.

[2] G. Arfken, H. Weber, and F. Harris. *Mathematical Methods for Physicists*, 7th Ed., Elsevier (2013), 375–377.

[3] E. Kreyszig, *Advanced Engineering Mathematics*, 7th Ed., Wiley (1993), 106–108.

[4] M.M. Guterman and C.H. Netecki, *Differential Equations – A First Course*, Saunders College Publishing (1988), 150–151.

[5] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer. (1999).

[6] B. J. Schroers , http://www.macs.hw.ac.uk/∼bernd/F13YB1/odenotes5.pdf.