Proving Ceva's Theorem with Isotropic Spinors

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Abstract

Ceva's Theorem is proved with Isotropic Spinors.

Introduction:
Euclidean geometry makes short work of this theorem. On the other hand, using spinors/vectors to solve it takes a bit more work. Vector methods seem unsophisticated by comparison. My methods may be naive, but they are sometimes interesting.

Ceva's Theorem
I'm not clever enough to put this theorem in words, so I'll just let the figure below and the following equation say it all:
In the figure below, line segments are taken from each vertex to the opposite side such that all three of them meet at point G. The following relation among the parts fo the sides of the triangle hold

$$\frac{EC}{BE}\frac{FA}{CF}\frac{DB}{AD} = 1\,. \tag{1}$$

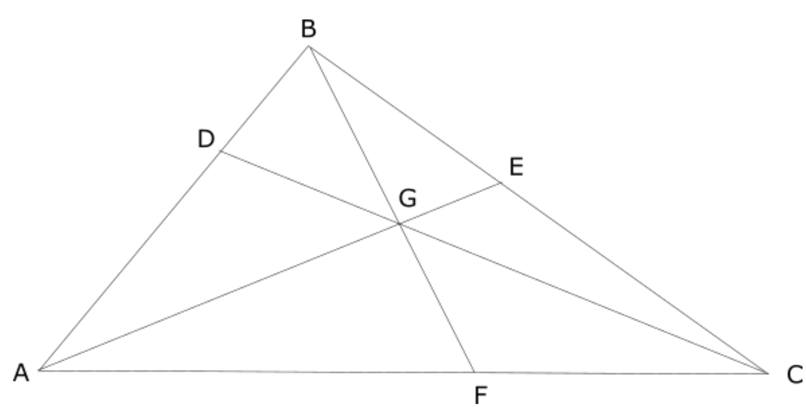

Figure 1. The Triangle for Ceva's Theorem.

Discussion: Generally speaking, in types of geometry problems such as this one, when using vector methods that stress the location of points and the differences between points (e.g., $A^\alpha - B^\alpha$), there rises two main categories of useful information. One is that of *collinearity* and the other of *concurrence*. In collinearity, we are interested in which points are constrained to lie on a given line. For concurrency: Three or more lines are said to be concurrent if they intersect in one point. Euclidean geometry and vector methods document this information in very different ways, and they utilize this information in very different ways, as well.

Let's take the current problem as a case in point (no pun intended). Take a standard triangle, such as Figure 1. Draw from each of two vertices (say A and C) line segments to their opposite sides. Naturally, they will meet in a point, which we'll label as G. Now, from the third vertex (B), draw a line segment through G until it meets its opposite side. Call that endpoint F, say. In Euclidean geometry we are done. But in a vector proof, we need to create a specific vector constraint that establishes that F really does reside **on** line segment $\overline{AC}$. Or, in other words, that points A, F, and C are collinear. The following is one of many way to do this using vector/spinor methods:

$$(C_\alpha - F_\alpha)(C^\alpha - A^\alpha) = 0\,. \tag{2}$$

You see, this sneaky symplectic product is really a cross product masquerading as an inner product. That's why in (2) the 'cross' product of vectors $C^\alpha - F^\alpha$ and $C^\alpha - A^\alpha$ is zero because the area of this degenerate parallelogram formed by collinear 'sides' is zero.

One result we will need in this proof is that the area of a parallelogram by two noncollinear vectors is the product of their lengths times the sine of the angle between them. For example, in the case of the parallelogram formed by nonparallel vectors $(C^\alpha - A^\alpha)$ and $(B^\alpha - A^\alpha)$, it area is given by

$$(C_\alpha - A_\alpha)(B^\alpha - A^\alpha) = AC \cdot AB \cdot \sin\angle A\,. \tag{3}$$

Proof:

Let's begin by constraining all these endponts of line segments going through the point G are on the sides of the triangle (these are the 'collinearity' constraints):

$$(A_\alpha - D_\alpha)(A^\alpha - B^\alpha) = 0\,, \tag{4a}$$
$$(B_\alpha - E_\alpha)(B^\alpha - C^\alpha) = 0\,, \tag{4b}$$
$$(C_\alpha - F_\alpha)(C^\alpha - A^\alpha) = 0\,. \tag{4c}$$

As a heuristic note: I've learned from experience that when I setup a system of similar constraints, that I should construct them similarly, that is, according to the same pattern, unless I have some *a priori* reason not to.

Now we express the concurrency constraints. Since the point G does not enter into the theorem claim in (1), we should choose the form of constraint equation that would allow us to easily get rid of the point G^α. To that end, we write them this way:

$$G^\alpha = A^\alpha + \sigma(E^\alpha - A^\alpha)\,, \tag{5a}$$
$$G^\alpha = B^\alpha + \mu(F^\alpha - B^\alpha)\,, \tag{5b}$$
$$G^\alpha = C^\alpha + \tau(D^\alpha - C^\alpha)\,, \tag{5c}$$

where σ, μ, and τ are scalar quantities to deal with – probably by simply eliminating them after we eliminate the G^α's. Let's begin by using the transitive property on (5a) and (5b), to get:

$$A^\alpha + \sigma(E^\alpha - A^\alpha) = B^\alpha + \mu(F^\alpha - B^\alpha)\,. \tag{6}$$

Subtracting B^α from both sides, we get

$$(A^\alpha - B^\alpha) + \sigma(E^\alpha - A^\alpha) = \mu(F^\alpha - B^\alpha)\,. \tag{7a}$$

Doing similarly for the other two pairs of equations we can form, we get

$$(A^\alpha - C^\alpha) + \sigma(E^\alpha - A^\alpha) = \tau(D^\alpha - C^\alpha)\,, \tag{7b}$$
$$(B^\alpha - C^\alpha) + \mu(F^\alpha - B^\alpha) = \tau(D^\alpha - C^\alpha)\,. \tag{7c}$$

At this point, we're finished with G, but are tasked with getting rid of these irksome scalars. Now, each of these equations can be scalarized to produce two equations with only one of these scalars per equation. That's six scalar equations in all, paired off to eliminate the scalars, as we did the point G^α. For example, from (7a), we get

$$(E_\alpha - A_\alpha)(A^\alpha - B^\alpha) = \mu(E_\alpha - A_\alpha)(F^\alpha - B^\alpha)\,, \tag{8a}$$
$$(F_\alpha - B_\alpha)(A^\alpha - B^\alpha) + \sigma(F_\alpha - B_\alpha)(E^\alpha - A^\alpha) = 0\,. \tag{8b}$$

From (7b), we get

$$(E_\alpha - A_\alpha)(A^\alpha - C^\alpha) = \tau(E_\alpha - A_\alpha)(D^\alpha - C^\alpha)\,, \tag{9a}$$
$$(D_\alpha - C_\alpha)(A^\alpha - B^\alpha) + \sigma(D_\alpha - C_\alpha)(E^\alpha - A^\alpha) = 0\,. \tag{9b}$$

And from (7c), we get

$$(F_\alpha - B_\alpha)(B^\alpha - C^\alpha) = \tau(F_\alpha - B_\alpha)(D^\alpha - C^\alpha)\,, \tag{10a}$$
$$(D_\alpha - C_\alpha)(B^\alpha - C^\alpha) + \mu(D_\alpha - C_\alpha)(F^\alpha - B^\alpha) = 0\,. \tag{10b}$$

On eliminating μ between (8a) and (10b), we get

$$\frac{(D_\alpha - C_\alpha)(F^\alpha - B^\alpha)}{(D_\alpha - C_\alpha)(B^\alpha - C^\alpha)}\,\frac{(E_\alpha - A_\alpha)(A^\alpha - B^\alpha)}{(E_\alpha - A_\alpha)(F^\alpha - B^\alpha)} = -1\,. \tag{11}$$

On eliminating σ between (8b) and (9b), we get

$$\frac{(F_\alpha - B_\alpha)(A^\alpha - B^\alpha)}{(F_\alpha - B_\alpha)(E^\alpha - A^\alpha)}\,\frac{(D_\alpha - C_\alpha)(E^\alpha - A^\alpha)}{(D_\alpha - C_\alpha)(A^\alpha - B^\alpha)} = 1\,. \tag{12}$$

Finally, on eliminating τ between (9a) and (10a), we get

$$\frac{(E_\alpha - A_\alpha)(A^\alpha - C^\alpha)}{(E_\alpha - A_\alpha)(D^\alpha - C^\alpha)} \frac{(F_\alpha - B_\alpha)(D^\alpha - C^\alpha)}{(F_\alpha - B_\alpha)(B^\alpha - C^\alpha)} = 1\,. \tag{13}$$

From here the way is clear. We multiply all three equations together and then hope for some cancellation of factors. Then we apply the as yet unuse collinearity conditions and hope for some more cancellations. Lastly, we convert what's left to area of parallelograms interpretations and see what's left after some more cancellations.

$$\begin{aligned} &\frac{(E_\alpha - A_\alpha)(F^\alpha - B^\alpha)}{(E_\alpha - A_\alpha)(A^\alpha - B^\alpha)} \frac{(D_\alpha - C_\alpha)(B^\alpha - C^\alpha)}{(D_\alpha - C_\alpha)(F^\alpha - B^\alpha)} \frac{(F_\alpha - B_\alpha)(A^\alpha - B^\alpha)}{(F_\alpha - B_\alpha)(E^\alpha - A^\alpha)} \times \\ &\qquad\times \frac{(D_\alpha - C_\alpha)(E^\alpha - A^\alpha)}{(D_\alpha - C_\alpha)(A^\alpha - C^\alpha)} \frac{(E_\alpha - A_\alpha)(A^\alpha - C^\alpha)}{(E_\alpha - A_\alpha)(D^\alpha - C^\alpha)} \times \\ &\qquad\times \frac{(F_\alpha - B_\alpha)(D^\alpha - C^\alpha)}{(F_\alpha - B_\alpha)(B^\alpha - C^\alpha)} = -1\,. \end{aligned} \tag{14}$$

Fortunately, we have some cancellations, leaving

$$\frac{(D_\alpha - C_\alpha)(B^\alpha - C^\alpha)}{(E_\alpha - A_\alpha)(A^\alpha - B^\alpha)} \frac{(F_\alpha - B_\alpha)(A^\alpha - B^\alpha)}{(D_\alpha - C_\alpha)(A^\alpha - C^\alpha)} \frac{(E_\alpha - A_\alpha)(A^\alpha - C^\alpha)}{(F_\alpha - B_\alpha)(B^\alpha - C^\alpha)} = 1\,. \tag{15}$$

At this point I'd like to replace these isotropic products by area replacements in the form of (3). Why? Because this is the bridge to go from the spinor products to the product of the lengths of line segments used in the theorem statement (1). There's only one problem left to overcome. A spinor product like $(D_\alpha - C_\alpha)(B^\alpha - C^\alpha) \mapsto DC \cdot BC \sin \angle C$, but there is no place for the lengths of the line segments going through the triangle, rather than lying on the triangle itself.

What I intend to do is to use an identity on the algebra of isotropic spinor to come to the rescue. I refer to it as *oscillation*. Here's how it works:

$$\begin{aligned} (D_\alpha - C_\alpha)(B^\alpha - C^\alpha) &= [(D_\alpha - B_\alpha) + (B_\alpha - C_\alpha)](B^\alpha - C^\alpha) \\ &= (D_\alpha - B_\alpha)(B^\alpha - C^\alpha)\,. \end{aligned} \tag{16}$$

The way I explain this is that the C_α was 'oscillated' to the B_α. I emphasize that this technique is not dependent on the problem at hand. And speaking of the problem at hand, the plan is to oscillate the vertex part of spinor difference where one spinor is a vertex and the other is a point on the opposite side, yielding

$$\frac{(D_\alpha - B_\alpha)(B^\alpha - C^\alpha)}{(E_\alpha - B_\alpha)(A^\alpha - B^\alpha)} \frac{(F_\alpha - A_\alpha)(A^\alpha - B^\alpha)}{(D_\alpha - A_\alpha)(A^\alpha - C^\alpha)} \frac{(E_\alpha - C_\alpha)(A^\alpha - C^\alpha)}{(F_\alpha - C_\alpha)(B^\alpha - C^\alpha)} = 1\,. \tag{17}$$

Now we're ready to convert to lengths of line segments:

$$\frac{DB \cdot BC \sin \angle B}{EB \cdot AB \sin \angle B} \frac{FA \cdot AB \sin \angle A}{DA \cdot AC \sin \angle A} \frac{EC \cdot AC \sin \angle C}{FC \cdot BC \sin \angle C} = 1\,. \tag{18}$$

Cancelling what we can, we get

$$\frac{DB}{EB}\frac{FA}{DA}\frac{EC}{FC} = 1\,, \tag{19}$$

which is equivalent to what we were to show. Okay, so where did we use the collinearity constraints? We did use them – implicitly when we converted the spinor products to the product of lengths of line segments and the sine of the angles between them. For example, for me to make the claim that

$$\frac{(D_\alpha - B_\alpha)(B^\alpha - C^\alpha)}{(E_\alpha - B_\alpha)(A^\alpha - B^\alpha)} \mapsto \frac{DB \cdot BC \sin \angle B}{EB \cdot AB \sin \angle B}\,, \tag{20}$$

assumes that D is on line segment $\overline{BA}$ and that E is on line segment $\overline{BC}$.