Control Bootcamp Notes for S. Brunton's Lecture Series (2017), Lectures 25–26

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Abstract

This paper contains my notes on Lectures 25–26 of Steve Brunton's 2017 presentation on Control Bootcamp, "Three Equivalent Representations of Linear Systems." These notes are meant to aid the viewer in following Brunton's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Three representations

It's time to move away from state space representations toward *transfer functions.*The state space representation is given by the system of equations

$$\begin{cases} \dot{x} &= Ax + Bu\,, \\ y &= cx\,. \end{cases} \tag{1}$$

In the figure below we have an abstract graphical representation of the system, which takes in inputs u and outputs y.

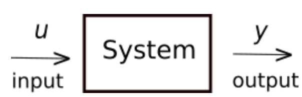

Figure 1. This is an abstract representation of the system and how it relates to its environment.

The mathematical representation of how the system converts an input to an output is called the *transfer function*, which is often represented by the letter G.

So, there are three equivalent ways to represent the transfer function for a linear system:

I: State Space (1).

II: The frequency domain $G(s)$, where s is a Laplace transform variable, and

$$G(s) = c(sI - A)^{-1}B\,. \tag{2}$$

III: The time domain representation or Impulse-Response, and

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau\,, \tag{3}$$

with $h(t)$ as the impulse.

By assuming linearity, if u_1 and u_2 are both inputs to the system and α is a scalar, then

$$\begin{aligned} u_1 + u_2 &\longrightarrow y_1 + y_2\,, \\ \alpha u + u_2 &\longrightarrow \alpha y\,, \end{aligned} \tag{4}$$

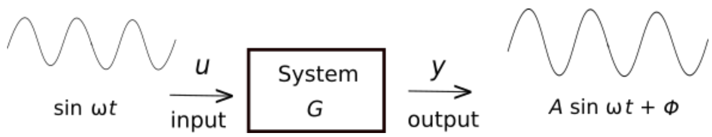

Figure 2. Sine wave in; modified sine wave out.

Note that the transfer function is complex. From which we have that the amplitude $A = |\, G(i\omega)\,|$ and the phase is given by $\angle G(i\omega) = \phi$.

2 Lecture 26: An example of frequency response

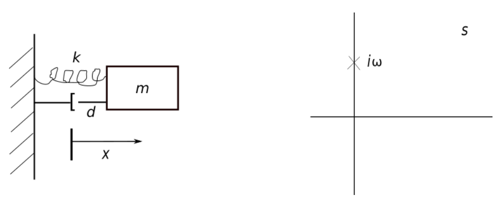

Figure 3. On the left is a Spring Mass Damper. On the right is the s complex plane.

2.1 Laplace Transforms

We shall represent the Laplace transform by the letter $\mathscr{L}$. Familiarity with Laplace transforms is assumed. From Laplace theory, we know that

$$\mathscr{L}\left\{\frac{dx}{dt}\right\} = s\overline{x}(s) - x(0)\,, \tag{5}$$

where $\mathscr{L}(x) = \overline{x}$. Equation (5) is the general rule for the transformation, but we will always assume that both initial conditions and transient effect are zero. Therefore, (5) becomes

$$\mathscr{L}\left\{\frac{dx}{dt}\right\} = s\overline{x}(s)\,. \tag{6}$$

For the second-order derivative we have

$$\mathscr{L}\left\{\frac{d^2x}{dt^2}\right\} = s^2\overline{x}(s)\,. \tag{7}$$

For a specific example, consider the case with $m = 1$, $d = 1$, and $k = 1$. Then our differential equation is

$$\ddot{x} + \dot{x} + x = u\,. \tag{8}$$

On taking the Laplace Transform of this, we get

$$s^2\overline{x} + s\overline{x} + \overline{x} = \overline{u}\,. \tag{9}$$

Factoring out $\overline{x}$, we get

$$(s^2 + s + 1)\overline{x} = \overline{u}\,. \tag{10}$$

At this point, we're interested in the ratio of $\overline{x}$ to $\overline{u}$, yielding

$$(s^2 + s + 1)\overline{x} = \overline{u}\,. \tag{11}$$

The ratio of $\overline{x}$ to $\overline{u}$ is a special function of s, called the *Tranfer Function*, ofteb expressed as $G(s)$. From the last equation, we get

$$G(s) = \frac{\overline{x}}{\overline{u}} = s^2 + s + 1\,. \tag{12}$$

From this we have the amplitude $|\,G(i\omega)\,| = A$, and phase $\angle G(i\omega) = \phi$.

2.2 Bode Plots

Do an experiment and take data. What follows is a Bode Plot.

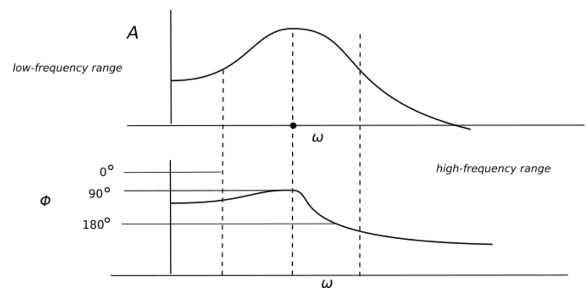

Figure 4. On the left is a Spring Mass Damper. On the right is the s complex plane.

The rest of this video concerned a Matlab demo.