Control Bootcamp Notes for S. Brunton's Lecture Series (2017), Lectures 27–30

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Abstract

This paper contains my notes on Lectures 27–29 of Steve Brunton's 2017 presentation on Control Bootcamp. These read-along notes are meant to aid the viewer in following Brunton's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 # 27 Control: Some review

It's time to move away from state space representations toward *transfer functions*. The state-space representation is given by the system of equations

$$\begin{cases} \dot{x} &= Ax + Bu\,, \\ y &= Cx\,. \end{cases} \tag{1}$$

In the figure below we have an abstract graphical representation of the system, which takes in inputs u and outputs y.

Figure 1. The system represented in the frequency domain, with frequency s.

Our first task is to Laplace transform the equations (1) to arrive at:

$$G(s) = c(sI - A)^{-1}B\,. \tag{2}$$

To accomplish this, let's first review the Laplace transform itself. We start with its definition

$$\mathscr{L}\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st}\,dt \equiv \overline{x}(s)\,. \tag{3}$$

We see that the effect of the integration is to integrate out the variable t and leave the variable s. This is the transformation from one variable to the other. Let's first compare this result to that of the Fourier transform.

$$\mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \equiv \bar{f}(\omega)\,. \tag{4}$$

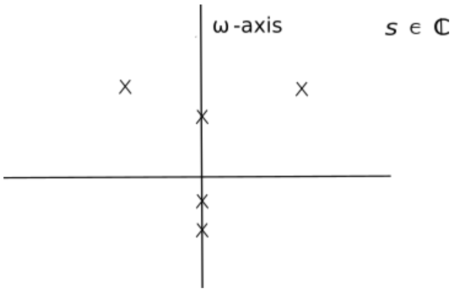

Figure 2. The vertical axis is imaginary and corresponds to the Fourier transform; whereas, the points off the vertical axis contain nonzero real components as well, allowing for factors of exponential increase or decrease.

Soon, we'll derive the transfer function:

$$G(s) = \frac{\bar{Y}(s)}{\overline{U}(s)} = c(sI - A)^{-1}B\,. \tag{5}$$

1.1 Laplace Transforms

We'll represent the Laplace transform by the letter $\mathscr{L}$. We begin with the Laplace transform of $\dot{x}$:

$$\begin{aligned}\mathscr{L}\{\dot{x}(t)\} = \mathscr{L}\left\{\frac{dx}{dt}\right\} &= \int_{0^-}^{\infty} \big(\frac{d}{dt}x\big)\, e^{-st}\, dt \\ &= e^{-st}x\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x e^{-st}\, dt \\ &= -x(0) + s\mathscr{L}\{x(t)\} \\ &= s\overline{x}(s) - x(0)\,,\end{aligned} \tag{6}$$

where, on the second step, we performed an integration by parts. On taking the Laplace transform of (1), we get

$$\begin{aligned} s\overline{x}(s) - x(0) &= A\overline{x}(s) + B\overline{u}(s)\,, \\ \overline{y}(s) &= c\overline{x}(s)\,, \end{aligned} \tag{7}$$

which can be rearranged to

$$\begin{aligned} (sI - A)\overline{x} &= B\overline{u}(s) + x(0)\,, \\ \overline{y}(s) &= c\overline{x}(s)\,. \end{aligned} \tag{8}$$

From this we get

$$\overline{x}(s) = (sI - A)^{-1}B\overline{u}(s) + (sI - A)^{-1}x(0)\,. \tag{9}$$

We note that $(sI - A)^{-1}$ will blow up at the eigenvalues of A. Anyway, we will simplify the last equation by setting $x(0) = 0$, which represents the steady-state equation. Then,

$$\overline{y}(s) = C(sI - A)^{-1}B\overline{u}(s) = G(s)\overline{u}(s)\,. \tag{10}$$

where $G(s)$ is the transfer function.

Now, if the function $u(t)$ is the impulse function $\delta(t)$ then $\overline{u}(s) = 1$. Therefore, the impulse response is the inverse Laplace transform of $G(s)$, that is, $\mathscr{L}^{-1}\{G(s)\}$.

2 Lecture # 28

Refer to Fig. 1.

The Benefits of Feedback to deal with:

- Stability (design eigenvalues)
- Uncertainty
- Disturbances

Example: Cruise control: We begin with a nominal model in which

$$\begin{aligned} &y = 2u\,, \\ &y \longleftrightarrow \text{ actual speed}\,, \\ &u \longleftrightarrow \text{ hitting accelerator / braking}\,. \end{aligned}$$

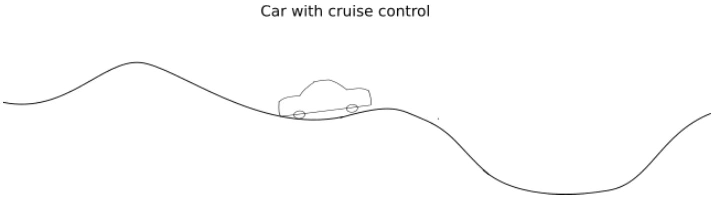

Figure 3. Car with cruise control must respond to the hilly terrain. Changes in the vertical present to us disturbances d.

2.1 Open-Loop vs. Closed-Loop Control

Our goal is: $y \to r$ "reference"

$$u_{\mathrm{OL}} = \frac{r}{2}\,. \tag{11}$$

If the actual system is different from the model, O.L. control cannot correct for that. For example, if $y = 2u$ but the actual car behaves as $y = u + d$, where d measures the disturbance to the system.

$$y_{\mathrm{OL}} = \frac{r}{2} + d\,, \tag{12}$$

which is only tracking half of the reference velocity.

To close the loop means to add some feedback into the architecture to deal with all our destablizing effects, as in the figure below:

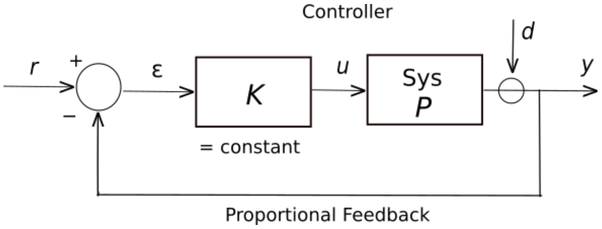

Figure 4. Closing the loop to add feedback control. Epsilon is the measured error.

What about for a closed-loop system? Now we compute the true y:

$$\begin{aligned} y &= u + d\,, && (13) \\ u_{\text{CL}} &= K\epsilon = K(r-y)\,, && (14) \\ y_{\text{CL}} &= Pu_{\text{CL}} + d = PKr - PKy_{\text{CL}} + d\,. && (15) \end{aligned}$$

From this we get

$$(1+PK)y_{\text{CL}} = PKr + d\,. \tag{16}$$

Solving this for y_{CL}, we have that

$$y_{\text{CL}} = \frac{PK}{1+PK}\,r + \frac{1}{1+PK}\,d\,. \tag{17}$$

Remember: The model has value $P = 2$ though the true value is $P = 1$.

Goal: Make $\dfrac{PK}{1+PK} \to 1$ and $\dfrac{1}{1+PK} \to 0$.

So, if the car speeds up due to going downhill, the additional speed it will gain will be measured and fed back into the circle to be subtracted-off. Our solution is to choose K large, but with caveat.

3 Lecture # 29

This lecture uses Matlab to review the equations of the last lecture.

4 Lecture # 30

Now, turning up K (the gain) will have the effect of making $PK/(1+PK)$ closer to 1, but

1) it can't every get to 1, and,

2) making K too large can cause instabililty.

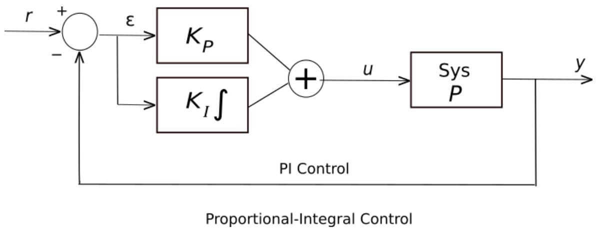

Figure 5. Proportional-Integral Control. The integral term reduces the steady-state error to zero.

4.1 Dynamics

Next, we need to add dynamics into the solution in state-space form:

$$\begin{cases} \dot{x} &= -x + u\,, \\ y &= x\,, \end{cases} \tag{18}$$

where x is the speed. Now, we perform a Laplace transformation.

$$\frac{\overline{y}(s)}{\overline{u}} = P(s) = \frac{1}{s+1}\,, \tag{19}$$

with a pole at $s = -1$. Real-life example:

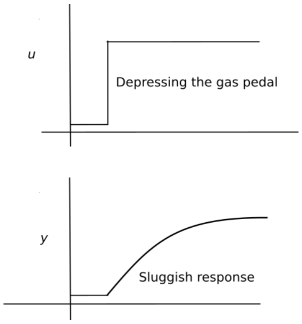

Figure 6. Sluggish response to depressing the gas pedal.

4.2 Introducing z

Our equation is

$$u = K_P(r-y) + K_I\int(r-y)\,dt\,. \tag{20}$$

But by making a variable substitution, we can eliminate the integral in favor of a couple of first-order differential equations.

$$\dot{z} = r - x\,, \tag{21a}$$
$$u = K_P(r-x) + K_I z\,. \tag{21b}$$

Or,

$$\dot{x} = -x - K_P x + K_I z - K_P r\,, \tag{22a}$$
$$\dot{z} = r - x\,. \tag{22b}$$

This can be put into matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} -1-K_P & K_I \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} -K_P \\ 1 \end{pmatrix} r\,, \tag{23}$$

$$y = (1\;0)\begin{pmatrix} x \\ z \end{pmatrix}\,. \tag{24}$$

And these equations can go into Matlab.