

PEMDAS and the Problem of Order of Operations

P. Reany

January 11, 2024

Abstract

There has been a long argument on the Internet that PEMDAS is not accurate for teaching the order of evaluations of operations of algebraic expressions in the real number system. I just want to put in my side of the issue.

1 Introduction

PEMDAS is a ‘modern’ way to determine the correct order of evaluation of terms and components in expressions involving real numbers and/or variables (indeterminants). It’s presented to primary/secondary school children. (The term is relatively new to me.) Clearly some system needs to be presented, but no scheme will work unless it 1) removes all ambiguity, 2) covers both displayed and in-line math presentation, and 3) provides rules for special (exceptional) cases. In fact, I dare say that no system of forming precedence rules will suit all cases, because people don’t like the use of many delimiters in expressions, and it’s delimiters that can remove ambiguity.

PEMDAS is the acronym for ‘Parentheses, Exponents, Multiplication, Division, Addition, Subtraction’.

Just so you’ll know, I didn’t learn PEMDAS when I went to school. Instead, I learned these rules:

Perform unary operations before binary operations.

The unary operators are the parentheses, absolute values, exponentiation, the negation (“-”), the affirmation (“+”), these latter indicating the sign of a number.

Generally speaking, evaluate parentheses contents first, then exponentiation, the negation and affirmation. Absolute value comes after parentheses.

The binary operations are multiplication, division, addition, and subtraction, to be performed in roughly that order. But, there is no real precedence between

multiplication and division.

Thus, if I encountered the expression

$$4 \div 2 \times 5 \div 9, \tag{1}$$

I would declare it as ill-formed and meaningless, and I'd tell whoever gave it to me to stick some parentheses into it to clarify it. However, as I interpret PENDINGAS, this expression could be rewritten as

$$((4 \div 2) \times 5) \div 9. \tag{2}$$

If you have a scientific calculator that understands PEMDAS, then use it, but be aware that the real world doesn't follow PEMDAS, and doesn't even consistently follow any logical set of rules.

2 The needed role of delimiters

Let's talk about the role of delimiters such as parentheses, square brackets, and curly brackets. This is where we should start. These are used to create an 'evaluation environment'. Every time you employ a pair of parentheses (open and closing), you create an evaluation environment. For example, consider

$$3 + (10x - 6y)/2 \tag{3}$$

We could interpret this as

$$(3 + (10x - 6y)/2) \tag{4}$$

in which case we evaluate the outer-most parentheses' contents first, which is the same as (3). Having done that, we can then evaluate the contents of the next level of nesting of delimiters, which is

$$10x - 6y \tag{5}$$

though this supposes we know the values of both x and y . Suppose we set them as $x = 3$ and $y = 2$, then

$$10x - 6y = 10 \cdot 3 - 6 \cdot 2 = 30 - 12 = 18, \tag{6}$$

and I demonstrated that one should apply multiplication before addition or subtraction. Then we get

$$3 + (18)/2 = 3 + 9 = 12. \tag{7}$$

And I divided before adding. So, that's following the naive rules and it will never be any simpler than that, but that is not enough. Consider the following example.

$$6(\frac{1}{2} + \frac{1}{3}) \tag{8}$$

Should we perform the evaluation inside the parentheses first or should we distribute the 6 over the summed terms inside? I'd prefer to distribute first, getting

$$6\left(\frac{1}{2} + \frac{1}{3}\right) = 6\left(\frac{1}{2}\right) + 6\left(\frac{1}{3}\right) = 3 + 2 = 5. \quad (9)$$

Of course it works, but it's not following the "rules."

I figured that my choosing to regard delimiters as unary operations, would be controversial, so I brought it to the attention of BingChat for its opinion, and it was decidedly against it.

On the other hand, however, ChatGPT 3.5 was more agreeable to my proposal:

Patrick:

I want to explain the order of precedence in the evaluation of operations on expressions in the real number system. I want to divide these operations into just two parts: unary and binary. Furthermore, I want the unary operations to include +, -, exponentiation, absolute values, and parentheses. Now, I know that typically parentheses are not included in this list, but if absolute values are unary operations, then parentheses should be considered to be as well, in my opinion. What do you think?

ChatGPT:

Your approach makes sense, and it's reasonable to consider parentheses as unary operations in this context. Parentheses, like absolute values, affect the evaluation of an expression by grouping elements and indicating a higher priority for the operations enclosed within them.

Here's a breakdown of the order of precedence for unary and binary operations in the real number system based on your criteria:

Unary Operations: 1. Exponentiation (e.g., raising a number to a power) 2. Absolute values 3. Parentheses (grouping operations) 4. Unary plus (+) and unary minus (-)

Binary Operations:

1. Multiplication and division (left to right) 2. Addition and subtraction (left to right) This order of precedence ensures that expressions are evaluated in a systematic way, taking into account the different levels of priority for both unary and binary operations. Keep in mind that exponentiation, absolute values, and parentheses take precedence over the basic arithmetic operations.

The binary operations are +, -, ×, /. The order of evaluation on these is: multiply and divide before add or subtract. If a division and multiplication meet, a syntax errors has occurred – the expression is not properly formed. The order

of evaluation between multiplication and division should always be established in the first place by use of delimiters such as parens, brackets, curly braces. When encountering parentheses (or some other delimiting pair), you start the whole evaluation process all over again and use something like recursion.

3 Examples from the Internet

I used one YouTube video, in particular, to inspire this essay on the order of operations in the algebra of real numbers.

The How and Why of Mathematics: PEMDAS is wrong.

<https://www.youtube.com/watch?v=1LCDca6dYpA>

However, PEMDAS has these rules for order of evaluation:

Parentheses, exponents, multiply/divide (left to right), then Add/Subtract (left to right).

Consider AB/C , which is not ambiguous because the only two reasonable interpretations are equal to each other:

$$AB/C = A \frac{B}{C} = \frac{AB}{C}. \quad (10)$$

In the context of Fourier analysis and Gaussians, I find the in-line expression

$$C(k) = e^{-k^2/4}/2\sqrt{\pi}, \quad (11)$$

which is displayed as

$$C(k) = \frac{e^{-k^2/4}}{2\sqrt{\pi}}, \quad (12)$$

and not as

$$C(k) = \frac{e^{-k^2/4}}{2} \sqrt{\pi}. \quad (13)$$

Technically, (12) is unambiguous, but (11) is ambiguous. Why didn't the author or editor of the paper make the expression unambiguous in the first place? I can think of two possible reasons. The first is that they prefer to keep as much mathematics "in-line" (that is, in the text part) as possible. And, second, that they prefer not to use clarifying parentheses, counting on the reader to perform the correct interpretation. Perhaps they think that too many parentheses are unesthetic.

In the context of GCD and LCD, we have a similar situation. Let m, n, r, s be integers. Then, an author writes mn/rs and hopes the reader will interpret this as either $(mn)/(rs)$, or in displayed for as¹

$$\frac{mn}{rs}. \quad (14)$$

¹I got this example from 'The How and Why of Mathematics', cited above.

My last example comes from beginning geometry. What is the formula for the area A of a triangle? In words, it's "one-half base times height." Let b = base length and h = height length. Then the formula has the displayed unambiguous form as

$$A = \frac{1}{2}bh. \tag{15}$$

But when this gets put in-line with text, it will be written in the forms

$$\frac{1}{2}bh \tag{16}$$

or

$$1/2bh \tag{17}$$

or

$$bh/2 \tag{18}$$

So, (16) is unambiguous but some authors/editors don't like it because it adds space between the lines before and after the line of text containing the expression. (17) is technically completely wrong, but often used (I suppose) on the assumption that the reader will interpret it correctly. (18) is technically completely correct, but is rarely used (I suppose), on the assumption that "non-functorial." That is, it's not being written down in the order it was spoken.

4 Conclusion

It seems that this argument will go on for a long time because, as I see it, there is no generally acceptable agreement on how to set the rules of precedence. On top of that, one can perform the operations contrary to the rules and still get the right answer. Thus, the precedence rules are, in the end, just guidelines for how to proceed, though some of the guidelines are more enforced than others.