# Teaching the HOW of Mathematics & Physics

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**Abstract**

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, January, 1995 (much shortened). Traditional approaches to teaching math and science largely emphasize facts to the exclusion of teaching how to think like a mathematician or a scientist. The key is to present the motivation and human reasoning that goes into these enquiries. This paper will look at this topic from the point of view of Einstein, G. Polya, and J. Willard Gibbs.

If there's anything that bugs me about how mathematics is taught is its presentation as a complete system, perfectly self-consistent with every last detail worked out, and with absolutely no controversy about it. Nothing could be further from the truth. And what is it that students need to learn under this false regime? Just a bunch of boring facts to apply to tedious problems. No wonder Americans hate mathematics! Formal education has reduced the most sublimely interesting field ever invented by man to mere praxis.

If we would give students motivated proofs, then many more of them would come to realize that proofs are not the mysterious things they believe them to be, and that even they can get in on the fun of inventing proofs to theorems. And this brings us to look at the styles of three highly respected mathematicians and physicists: A. Einstein, J. Willard Gibbs, and G. Polya. Let's do Polya first.

In December, 1949 G. Polya had an article published in the AMM, "With, or without, motivation?" It's an important and, I'm sure, little-read note about writing proofs in an informal and motivational approach. He gave two very different proofs of a theorem in analysis as an example.

**Theorem:**

*If the terms of the sequence $a_1, a_2, a_3, \cdots$ are nonnegative real numbers, not all equal to zero, then*

$$\sum_1^\infty (a_1 a_2 a_3 \cdots a_n)^{1/n} < e \sum_1^\infty a_n\,.$$

proof (unmotivated): Define the numbers $c_1, c_2, c_3, \cdots$ by

$$c_1 c_2 c_3 \cdots c_n = (n+1)^n$$

for $n = 1, 2, 3, \cdots$. We use this definition, then the inequality between the arithmetic and the geometrical means, and finally that fact that the sequence defining $e$, the general term of which is $[(k+1)/k]^k$, is increasing. We obtain

$$\begin{aligned}
\sum_1^\infty (a_1 a_2 a_3 \cdots a_n)^{1/n} &= \sum_1^\infty \frac{(a_1 c_1 a_2 c_2 \cdots a_n c_n)^{1/n}}{n+1} \\
&\le \sum_1^\infty \frac{a_1 c_1 + a_2 c_2 + \cdots + a_n c_n}{n(n+1)} \\
&= \sum_{k=1}^\infty a_k c_k \sum_{n \ge k}^\infty \frac{1}{n(n+1)} \\
&= \sum_{k=1}^\infty a_k c_k \sum_{n=k}^\infty \left( \frac{1}{n} - \frac{1}{n+1} \right) \\
&= \sum_{k=1}^\infty a_k \frac{(k+1)^k}{k^{k-1}} \frac{1}{k} \\
&< e \sum_{k=1}^\infty a_k \,.
\end{aligned}$$

I have given the proof in its entirety to fairly represent it. Polya goes on to strongly deride such a presentation because of its unmotivated first step that brings-in the numbers $c_k$ from the blue, saying: "Perhaps the author knows the purpose of this step, but I do not and, therefore, I cannot follow him with confidence."

Polya derides this formal approach as egotistical, in my opinion. His own motivated proof which he then gives is too long to reproduce here. Polya defends his version of the proof on the basis that it not only teaches the steps it also teaches how they were arrived at—by false starts and near solutions that were turned into full solutions. The point is that it teaches HOW to do proofs, i.e, the heuristics. He concludes (p.690–691):

> Now I come to my point: I think that also heuristic logic is closely connected with mathematics, but not with mathematical theories and their deductive structure, rather with mathematical problems and the invention of their solution. In fact, I think that heuristic logic could make serious progress in studying such plausible motives of the solution as were emphasized in the long presentation of our example.

What a tragedy we have here. For three centuries mathematicians have been throwing away the heuristics to their proof, thus denying future generations the heuristics that should have been their legacy to inherit. But now it's all wasted, thrown away into the garbage heap of egotistical loss and formalist rationalization. And we continue to throw away these insights today to be lost forever. When will this waste be stopped?

I wish to offer a similar example of motivated and unmotivated proofs to the following problem:

*The equations of motions for a projectile under quadratic drag with a nearly horizontal trajectory can be approximated by the couple:*

$$\dot{v}_\perp = -\alpha {v_\perp}^2 \,, \tag{1a}$$

$$\dot{v}_\parallel = g - \alpha v_\perp v_\parallel \,. \tag{1b}$$

where $v_\parallel$ is the component of velocity in the direction of the local gravitational force, and with $r_{0\parallel} = r_{0\perp} = 0$.

*Using (1a) and (1b) as givens, show that the solutions for $r_\perp$ and $r_\parallel$ are given by*

$$r_\perp = \frac{1}{\alpha}\log\left(\alpha t v_{0\perp} + 1\right),$$

$$r_\parallel = \tfrac{1}{4} g t^2 + \frac{gt}{2\alpha v_{0\perp}} + \left(\frac{v_{0\parallel}}{\alpha v_{0\perp}} - \frac{g}{2\alpha^2 {v_{0\perp}}^2}\right)\log\left(\alpha v_{0\perp} t + 1\right).$$

**Proof:** From (1a) we quickly convert to get

$$D_t(v_\perp^{-1}) = \alpha \,, \tag{2}$$

and integrate to get

$$v_\perp^{-1} = {v_{0\perp}}^{-1} + \alpha t \,. \tag{3}$$

Therefore,

$$r_\perp = \frac{1}{\alpha}\log\left(\alpha t v_{0\perp} + 1\right). \tag{4}$$

Now, let $\omega = v_\parallel v_\perp^{-1}$. (Why do this? Because we search for ways to relate the derivatives of $v_\parallel$ and $v_\perp$ and this trick worked for me.) Then,

$$\begin{aligned}\dot{\omega} &= \dot{v}_\parallel v_\perp^{-1} + v_\parallel \, D_t(v_\perp^{-1}) \\ &= (g - \alpha v_\perp v_\parallel) v_\perp^{-1} + \alpha v_\parallel \\ &= g v_\perp^{-1} \\ &= g(\alpha t + {v_{0\perp}}^{-1}) \,, \end{aligned} \tag{5}$$

where we used (3) to get to the last step. (Note that $\ddot{\omega}$ is a constant of motion.) Integrating this gives

$$\omega = v_\parallel v_\perp^{-1} = v_{\parallel 0} v_{\perp 0}^{-1} + \tfrac{1}{2} g \alpha t^2 + g {v_{0\perp}}^{-1} t \,. \tag{6}$$

From this we get

$$v_\parallel = \left[\frac{v_{0\parallel}}{v_{0\perp}} + \tfrac{1}{2} g \alpha t^2 + g {v_{0\perp}}^{-1} t\right] \frac{v_{0\perp}}{\alpha t v_{0\perp} + 1} \,,$$

which can be integrated directly to get the required answer. The trick to these tricky integrals is either to use integral tables or, if you're going to do them without tables, to use both additive and multiplicative virtual emplacements. For example,

$$\begin{aligned} g \int \frac{t\,dt}{\alpha t v_{0\perp} + 1} &= \frac{g}{(\alpha v_{0\perp})^2} \int \frac{\alpha t v_{0\perp}\, d(\alpha t v_{0\perp} + 1)}{\alpha t v_{0\perp} + 1} \\ &= \frac{g}{(\alpha v_{0\perp})^2} \left[ \int \frac{(\alpha t v_{0\perp} + 1)\, d(\alpha t v_{0\perp} + 1)}{\alpha t v_{0\perp} + 1} - \int \frac{d(\alpha t v_{0\perp} + 1)}{\alpha t v_{0\perp} + 1} \right] \\ &= \frac{gt}{\alpha v_{0\perp}} - \frac{g}{(\alpha v_{0\perp})^2} \log\left(\alpha t v_{0\perp} + 1\right) . \end{aligned} \tag{7}$$

The reader can easily follow the steps—that's not the issue. The issue is how I knew to let $\omega = v_{\parallel} v_{\perp}^{-1}$ and then to differentiate it. According to the formalist viewpoint, the proof is complete as is, but not to me. The reason for choosing $\omega$ as I did is two-fold. The first reason is from Equation (2), which offers the hope of a simplification (which occurred when it was substituted into (5)). And second, although the analogy is not perfect there is an apparent similarity between this present case and the Wronskian of the solutions to a 2nd-order differential equation with solutions $y_1, y_2$:

$$W = y_1^2 \frac{d}{dx} \left( \frac{y_2}{y_1} \right) . \tag{8}$$

Based on this shaky analogy and on many others from the general theory, it's reasonable to at least try the derivative of the ratio of the two solutions. Fortunately, it paid off in this case. But the attempt was not an out-of-the-blue miracle.

Note: We can regard the integral of the derivative of $\omega$ given before as a virtual emplacement

$$\omega - \omega_0 = \int d(\omega - \omega_0) = \int \frac{d}{dt} (\omega - \omega_0)\, dt = \int \dot{\omega}\, dt .$$

So what general heuristic rule can be derived from this particular example? Well, that when a pattern shows up more than once in a general field then expect that it will come up in other ways as well.

Let's go on now to J. Willard Gibbs, the influential physicist that "invented" modern vector calculus and also laid the foundation to modern thermodynamics. I'm getting my information about Gibbs from a September-1990 article by Martin J. Klein (p. 40–48) from *Physics Today.* Though it was Clausius who had first defined entropy in thermodynamics, its importance was not realized until Gibbs introduced the equation

$$dU = TdS - PdV . \tag{9}$$

This formulation of thermodynamics was more suitable to Gibbs's geometrical approach to the subject. In his 1876 and 1878 works entitled "On the Equilibrium of Heterogenous Substances," Gibbs introduced chemical potential and the phase rule, all in all, vastly extending the scope of thermodynamics to include "elastic, surface and electrochemical phenomena by a single, unified method" (p.44). He went on to say on the same page:

> As Pierre Duhem once remarked, it took "a remarkable perspicacity" on the part of J. D. van der Waals, who first saw its power, to perceive the phase rule "among the algebraic formulas where Gibbs had to some extent hidden it." And Duhem wondered how many more such seeds that might have grown into whole programs of research "had remained sterile because no physicist or chemist had noticed them under the algebraic shell that concealed them?"

Unlike Gibbs, who believed that the formal presentation is all that should be given, Einstein believed that presenting the motivation was crucial. It is likely that he wrote more essays and gave more lectures clarifying expositions on his relativity theory than any other physicist did on his or her pet theories.

I'm not saying that the right place to present one's entire line of thought is in a technical journal, but I do think that if that line of thought was not obvious it should be recorded for the rest of us to learn from.