Fireside Treatise on Virtual Emplacement, Part 2

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February 16, 2020

Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, January, 1996. Virtual emplacement is a means to change the form of an expression or equation by familiar identity operations, leading to an improvement in the situation.

Introduction

The subject of virtual emplacement (VE) has been brought up many times before in this journal. The purpose of the constant re-exposure is to facilitate "seeing" where to use VEs in a given domain of mathematics by learning all the ways they have been used before. This part of the series is the continuation from the April 1992 article in this journal.

A virtual emplacement is defined as the changing of the form of a mathematical expression or equation without changing its value. Technically speaking, this is no more than just using an identity, but if that was all there is to it, we'd all be great mathematicians. The trick, of course, is not to be able to recognize a VE when we see it, though that's not always as easy as it may first seem, but to know when to use it when you don't see it.

An example should help. Show that

$$\lim_{h\to 0}(1+h)^{1/h} = e\,. \tag{1}$$

The question is, Where does the e come from?! Who knows, but we can put it there virtually.

$$\begin{aligned}\lim_{h\to 0}(1+h)^{1/h} &= \lim_{h\to 0}\exp\left\{\ln(1+h)^{1/h}\right\}\\ &= \exp\left\{\lim_{h\to 0}\frac{1}{h}\ln(1+h)\right\} = \exp\{1\} = e\,,\end{aligned} \tag{2}$$

where we used l'Hopital's Rule.

With all due respect to the need for rigor in mathematics, that's no excuse for killing off virtually all interest mathematics it in the U.S. to high-school and college students.

From vector spaces:

In a linear vector space $\mathcal{V}$ we know by its defining axiom that every element v in $\mathcal{V}$ has an additive inverse element, v' say, such that $v + v' = 0$. Now, show that v' is unique.

Proof:

Assume there exists some element v'' of $\mathcal{V}$ possibly different from v' such that $v + v'' = 0$. Now

$$v'' = v'' + 0 = v'' + (v + v') = (v'' + v) + v' = 0 + v' = v' \,. \tag{3}$$

Thus v'' is v' afterall. The additive inverse is unique.

From complex analysis:

If we divide one complex number by another, and put the result in the form $u + iv$, what do we get?

$$\frac{c+id}{a+ib} = \frac{c+id}{a+ib}\frac{a-ib}{a-ib} = \frac{(ac+bd)}{a^2+b^2} + i\,\frac{(ad-bc)}{a^2+b^2}\,. \tag{4}$$

Now, show that

$$1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \cdots = \frac{1-a\cos\theta}{1-2a\cos\theta+a^2} \tag{5}$$

where $|\,a\,| < 1$.

Proof:
We use that

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}\,, \tag{6}$$

when $|\,z\,| < 1$. Now let $z = a\,e^{i\theta}$, then

$$1 + ae^{i\theta} + a^2e^{2i\theta} + a^3e^{3i\theta} + \cdots = \frac{1}{1-ae^{i\theta}}\,. \tag{7}$$

Thus

$$\begin{aligned}(1 + a\cos\theta + &a^2\cos 2\theta + a^3\cos 3\theta + \cdots) + i(\cdots)\\ &= \frac{1}{1-ae^{i\theta}}\frac{1-ae^{-i\theta}}{1-ae^{-i\theta}} = \frac{1-a\cos\theta+ia\sin\theta}{1-2a\cos\theta+a^2}\,.\end{aligned} \tag{8}$$

Equation (5) is just the real part of equation (8).

Cauchy's Integral Formula:

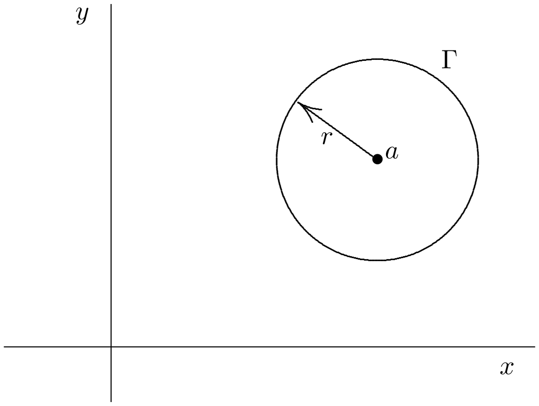

Figure 1

Let $f(z)$ be analytic inside and on the boundary $\mathcal{C}$ of a simply connected region $\mathcal{R}$ containing the point a. Prove Cauchy's Integral formula:

$$f(a) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z-a}\, dz \tag{9}$$

By a well-known theorem in complex analysis we can replace the loop integral over the curve $\mathcal{C}$ by the loop integral over an interior circle Γ centered at point a, as in Fig. 1.

Thus

$$\begin{aligned} \oint_{\Gamma} \frac{f(z)}{z-a}\, dz &= \oint_{\Gamma} \frac{f(z)-f(a)}{z-a}\, dz + \oint_{\Gamma} \frac{f(a)}{z-a}\, dz \\ &= \oint_{\Gamma} \frac{f(z)-f(a)}{z-a}\, dz + 2\pi i f(a) \end{aligned} \tag{10}$$

But

$$\oint_{\Gamma} \frac{f(z)-f(a)}{z-a}\, dz = \oint_{\Gamma} f'(a)\, dz + \oint_{\Gamma} \mu\, dz = \oint_{\Gamma} \mu\, dz \tag{11}$$

The result follows after using standard arguments to show that the last term is also zero (i.e., take the radius of the circle to approach zero).

Derivatives:

Let $f(z)$ be analytic inside and on the boundary $\mathcal{C}$ of a simply connected region $\mathcal{R}$, then

$$f'(a) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z-a)^2}\, dz \tag{12}$$

Similar to our approach in the last problem, we take a and $a+h$ to lie in $\mathcal{R}$, which contains the circle Γ. Also from the last problem we can write that

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{1}{h}\left\{\frac{1}{z-(a+h)}-\frac{1}{z-a}\right\} f(z)\,dz \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)dz}{(z-a-h)(z-a)} \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)dz}{(z-a-h)(z-a)}\,\frac{z-a}{z-a} \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)dz}{(z-a-h)(z-a)}\,\frac{(z-a)-h+h}{z-a} \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)dz}{(z-a-h)(z-a)}\,\frac{(z-a-h)+h}{z-a} \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)dz}{(z-a)^2}+\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{h\,f(z)\,dz}{(z-a-h)(z-a)^2}\,. \qquad (13)
\end{aligned}$$

The result follows on taking $h \to 0$ and then taking the loop integral over a circle Γ and taking its radius to zero. Very beautiful virtual emplacements!

Liouville's theorem

Let $f(z)$ be analytic and bounded on the entire complex plane, then $f(z)$ is a constant.

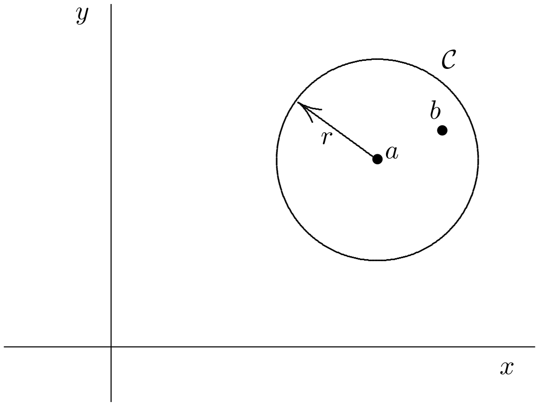

Figure 2

Let a and b be given as in Fig. 2, then by Cauchy's integral formula we have

$$\begin{aligned}
f(b)-f(a) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z-b}\,dz-\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z-a}\,dz \\
&= \frac{b-a}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)dz}{(z-b)(z-a)} \qquad (14)
\end{aligned}$$

But $|\,z-a\,| = r$ and

$$|\,z-b\,| = |\,z-a+a-b\,| \geq |\,z-a\,| - |\,a-b\,| = r - |\,a-b\,| \geq r/2 \qquad (15)$$

by choosing the placement of b such that $r > 2|\,a-b\,|$. Let M be the maximum value of $f(z)$ on $\mathcal{C}$. Thus,

$$|\,f(b)-f(a)\,| = \frac{|\,b-a\,|}{2\pi}\left|\oint_{\mathcal{C}} \frac{f(z)dz}{(z-b)(z-a)}\right| \leq \frac{|\,b-a\,|M(2\pi r)}{2\pi(r/2)r} = \frac{2|\,b-a\,|M}{r}\,. \qquad (16)$$

So, on letting $r \to \infty$ we get $f(a) = f(b)$, thus $f(z)$ is a constant.

Derive Poisson's integral formulas for the half plane.

Consider a circle $\mathcal{C}$ of radius R centered at the origin and containing the point λ in the upper half plane above the real axis. Now restrict this closed loop to just the upper half circle and the real axis from $-R$ to R. Let $f(z)$ be analytic on the upper-half plane and goes to infinity no faster than R as $|z| \to \infty$.

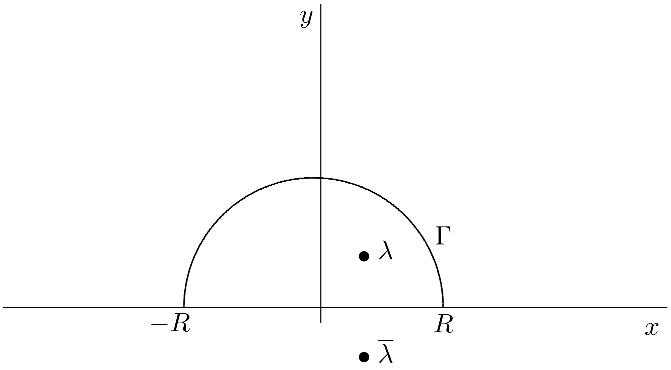

Figure 3

By Cauchy's integral formula we have

$$f(\lambda) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z-\lambda}\,dz\,, \qquad 0 = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z-\overline{\lambda}}\,dz\,. \qquad (17)$$

By virtually emplacing the second into the first we have

$$f(\lambda) + 0 = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z-\lambda}\,dz - \frac{f(z)}{z-\overline{\lambda}}\,dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{(\lambda-\overline{\lambda})f(z)\,dz}{(z-\lambda)(z-\overline{\lambda})} \qquad (18)$$

On writing λ as $\mu + i\nu$, where μ and ν are real, we get

$$f(\lambda) = \frac{1}{\pi}\int_{-R}^{R} \frac{\nu f(x)\,dx}{(x-\mu)^2+\nu^2} + \frac{1}{\pi}\oint_{\Gamma} \frac{\nu f(z)\,dz}{(z-\lambda)(z-\overline{\lambda})} \qquad (19)$$

where Γ is the semi-circular arc above the real axis. The second term goes to zero as $R \to \infty$, leaving just

$$f(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\nu f(x)\, dx}{(x-\mu)^2 + \nu^2} \tag{20}$$

The zeta function:

Show that the zeta function

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} \tag{21}$$

is analytic in the region of the z-plane where $\mathrm{Re}(z) \geq 1+\delta$, where δ is any fixed positive number. We use the Weierstrass M test. First, each term of the series is an analytic function. Second, for $x = \mathrm{Re}(z) \geq 1+\delta$

$$\left| \frac{1}{k^z} \right| = \left| \frac{1}{e^{z \ln k}} \right| = \frac{1}{e^{x \ln k}} = \frac{1}{k^x} \leq \frac{1}{k^{1+\delta}} \tag{22}$$

And since $\sum 1/k^{1+\delta}$ converges, by the Weierstrass M test, $\sum_{k=1}^{\infty} 1/k^{1+\delta}$ converges uniformly for $\mathrm{Re}(z) \geq 1+\delta$. Thus $\zeta(z)$ is analytic in this region.