Fireside treatise on virtual emplacement, Part 1

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March 17, 2025

Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, July, 1992. It introduces the art of virtual emplacement, and demonstrates it with many examples. Though some of the later examples are difficult, this article is meant to be a source of both instruction and enjoyment in its review of the power of such a simple concept as virtual emplacement.

> **You have to know what to look for so you can spot it.**
> —Papago Indian drug-enforcement border scout.

This article is meant to be both instructional and entertaining, and it extends the treatment started in the last issue. The *method of virtual emplacement* is the transformation of the form of a mathematical expression, while leaving its value unchanged.

Although *virtual emplacement (VE)* is nothing more than using identities, it remains a powerful and surprisingly all-too-easy identity to overlook until you see it done by someone else. My belief is that if the reader will study the examples herein with an eye for patterns of the use of VE, he or she will develop a knack for using it in creative and powerful ways.

I will be presenting examples in order of easiest to most difficult areas of math, so don't be alarmed if you can't follow all the examples at this stage of your mathematical maturity. You can always refer back to it later on.

Modulo arithmetic:

If $y \equiv 0 \mod r$ then $x \mod r$ can be replaced by $(x+y) \mod r$ without changing its value.

Algebra:

Perhaps our first introduction to VE occurs in algebra. Two instances stand out right away. They are that you can add 0 to any number or expression and not change its value, and that you can multiply any number or expression by unity and not change its value. Interestingly, these two types of VEs continue to be the dominant forms of VEs throughout mathematics until the introduction of grade-selection in complex and Clifford algebras, which we will get to later.

As an example, consider this familiar problem: Find a single fraction to represent $\frac{1}{3}+\frac{1}{7}$.

$$\frac{1}{3}+\frac{1}{7}=\frac{1}{3}\cdot 1+\frac{1}{7}\cdot 1=\frac{1}{3}\cdot\frac{7}{7}+\frac{1}{7}\cdot\frac{3}{3}=\frac{10}{21}\,. \tag{1}$$

We can see in this example another common form of VE, which we'll call the "do-and-undo" VE. Replacing 1 by $\frac{3}{3}$ is multiplying and dividing the number 1 by 3. This form of VE takes its best examples in mixing logs with exponentials, squares with square roots, etc.

As an example, consider the task of writing $\sqrt{x}+\sqrt{y}$, as a dependent on z and w with $z=x+y$, $w=\sqrt{xy}$, and w,x,y,z positive reals.

$$\sqrt{x}+\sqrt{y}=\sqrt{\left[\sqrt{x}+\sqrt{y}\right]^2}=\sqrt{x+2\sqrt{xy}+y}=\sqrt{z+2w}\,. \tag{2}$$

This next technique, called *rationalizing a fraction*, crops up frequently. The basic idea is to clear either the numerator or denominator of radicals. In algebra we learn that writing $1/\sqrt{2}$ is a scorned act. We are told that we *must* rationalize this:

$$\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2} \tag{3}$$

Great, the world is once again safe for rational thought. It is sufficient to know that these are equivalent expressions.

A little more complicated is the following: For a,b distinct, positive reals

$$\frac{1}{\sqrt{a}+\sqrt{b}}=\frac{1}{\sqrt{a}+\sqrt{b}}\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{a}-\sqrt{b}}{a-b} \tag{4}$$

Furthermore, we have one of the most prevalent of VEs: For arbitrary reals A,B,

$$|B|=|(B-A)+A|\le|B-A|+|A|\,. \tag{5}$$

In the so-called "completing the square" we look for a means to replace the expression x^2+ax by $(x+\lambda)^2$. With $\lambda=a/2$ we have

$$x^2+ax=x^2+ax+\tfrac{1}{4}a^2-\tfrac{1}{4}a^2=(x+a/2)^2-\tfrac{1}{4}a^2\,. \tag{6}$$

The problem here is that we didn't replace x^2+ax by $(x+\lambda)^2$, just as in real life when you can't always get exactly what you want or exactly what you're told to do. What we did do is to go for the *next best thing.* If you can't get the best to be hoped for, go for the *next best thing (NBT).* Though, in reality, there will likely be more than just one next best thing, context and personal choice will select a single one.

I'll finish this part with examples from even/odd and periodic functions. If $f(x)$ is even (odd) then $f(x) = f(-x)$ ($f(x) = -f(-x)$). If $f(x)$ is periodic with period τ then $f(x) = f(x + \tau)$.

Plane geometry:

This form of mathematics does not produce VEs in the abundance that algebra does. About all I can think of now is that of the auxiliary line, which is kind of a NBT to a "real" example.

But if you use vectors to do plane geometry, you can create a triangle of vectors out of a single vector by a simple VE. Consider the vector in the plane given by $\mathbf{a} - \mathbf{b}$, which is a vector with base at $\mathbf{b}$ and tip at $\mathbf{a}$. Now add to this mix any point $\mathbf{c}$ in the plane not on the line generated by the vector $\mathbf{a} - \mathbf{b}$ and we get a vector equation identifying the sides of triangle $\mathbf{abc}$:

$$\mathbf{a} - \mathbf{b} = (\mathbf{a} - \mathbf{c}) + (\mathbf{c} - \mathbf{b})\,. \tag{7}$$

Trigonometry:

I could spend a lot of time on this section, but I think just a few examples will be representative. We already mentioned even and odd and periodic functions, so let's see some examples:

$$\begin{aligned} &\cos x = \cos(-x)\,, \qquad \sin x = -\sin(-x) \\ &\cos x = \cos(2\pi k + x)\,, \qquad \sin x = \sin(2\pi k + x)\,. \end{aligned} \tag{8}$$

And as an example:

$$\tan x = \frac{\sin x}{\cos x} = \frac{-\sin(x + \pi)}{-\cos(x + \pi)} = \tan(x + \pi)\,. \tag{9}$$

Calculus:

Calculus is loaded with VEs. It's hard to know where to start. Let's start with this problem: Show that $\lim_{n\to\infty} \sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big) = 1/2$.

$$\begin{aligned} \lim_{n\to\infty} \sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big) &= \lim_{n\to\infty} \left[\sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right] \\ &= \lim_{n\to\infty} \left[\frac{\sqrt{n}(n + 1 - n)}{\sqrt{n+1} + \sqrt{n}}\right] \\ &= \lim_{n\to\infty} \left\{\left[\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right] \frac{1/\sqrt{n}}{1/\sqrt{n}}\right\} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{(1 + 1/n)} + 1} = \frac{1}{2}\,. \end{aligned} \tag{10}$$

This is a good place to mention that delimiters such as (,) and [,] can be used as VEs, as the brackets were used in this problem.

Now for the chain rule. Let $u = u(t)$ be a differentiable function of t at t_0, and let $z = z(u)$ be a differentiable function of u at $u_0 = u(t_0)$, then $dz/dt = (dz/du)(du/dt)$ where the derivatives are evaluated at $t = t_0$.

$$\frac{dz}{dt} = \lim_{h\to 0} \frac{z(t_0+h) - z(t_0)}{h} \tag{11}$$

$$= \lim_{h\to 0} \frac{z(u(t_0+h)) - z(u(t_0))}{u(t_0+h) - u(t)} \cdot \frac{u(t_0+h) - u(t)}{h} \qquad \text{(note VE)} \tag{12}$$

$$= \lim_{h\to 0} \frac{z(u_0 + hu'(t_0)) - z(u_0)}{hu'(t_0)} \lim_{h\to 0} \frac{u(t_0+h) - u(t)}{h} = \frac{dz}{du} \cdot \frac{du}{dt}\,, \tag{13}$$

where we used that as $h \to 0$ so does $hu'(t_0)$.

Let's do something similar to the above. With respect to a standard coordinate system, if the horizontal speed of a baseball is $\dot{x} = 80\,\mathrm{m/s}$ and its vertical speed is $\dot{y} = 17\,\mathrm{m/s}$, what is the angle of elevation of the ball? Solution: The ball is assumed to travel a smooth curve through space, approximately in a vertical plane. Now dy/dx is the instantaneous slope of the tangent line to the curve. Thus,

$$\tan\theta = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} = \frac{17}{80}\,. \tag{14}$$

So $\theta = \tan^{-1} 17/80$.

No treatment of calculus would be complete without a bout with δ, ϵ stuff. The last exercise used the fact that the limit of a product is the product of the limits, so to speak. We will show how this is proved for the similar case of the limit of a quotient (see Taylor/Mann, 1972, 72-4). Let f and g be defined on an interval containing $x = x_0$, but not necessarily at the point x_0 itself. If both $\lim\limits_{x\to x_0} f(x)$, $\lim\limits_{x\to x_0} g(x)$ exist, show that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}\,. \tag{15}$$

Proof: Let the limits of $f(x)$ and $g(x)$ be represented by A and $B (\neq 0)$, respectively. Now, since $\lim_{x\to x_0} g(x) = B$, we know that $|g(x)| > \frac{1}{2}|B|$ for x sufficiently near to x_0. Thus we choose δ_0 so that

$$|g(x) - B| < \tfrac{1}{2}|B| \quad \text{if} \quad 0 < |x - x_0| < \delta_0\,. \tag{16}$$

Then, since $B = (B - g(x)) + g(x)$ (our first VE here), and thus

$$|B| \leq |B - g(x)| + |g(x)| < \tfrac{1}{2}|B| + |g(x)|\,, \tag{17}$$

then,

$$\tfrac{1}{2}|B| < |g(x)| \quad \text{if} \quad 0 < |x - x_0| < \delta_0\,. \tag{18}$$

Now we can write

$$\frac{f(x)}{g(x)} - \frac{A}{B} = \frac{Bf(x) - Ag(x)}{g(x)B} = \frac{B[f(x) - A] + A[B - g(x)]}{g(x)B}\,, \tag{19}$$

where we used another VE. Thus,

$$\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| \leq \frac{|B||f(x) - A| + |A||B - g(x)|}{|g(x)||B|}\,. \tag{20}$$

Now with ϵ arbitrary, we look for a δ so that

$$\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| \leq \epsilon \quad \text{if} \quad 0 < |x - x_0| < \delta_0\,. \tag{21}$$

To that end, let

$$\epsilon_1 = \frac{|B|^2 \epsilon}{2(|B| + |A|)}\,, \tag{22}$$

then, choose δ_1 and δ_2 such that

$$|f(x) - A| < \epsilon_1 \quad \text{if} \quad 0 < |x - x_0| < \delta_1\,, \tag{23}$$

$$|g(x) - B| < \epsilon_1 \quad \text{if} \quad 0 < |x - x_0| < \delta_2\,, \tag{24}$$

with $\delta_2 < \delta_0$. Thus

$$\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| \leq \frac{|B|\epsilon_1 + |A|\epsilon_1}{\frac{1}{2}|B|^2} \equiv \epsilon\,, \tag{25}$$

whenever $0 < |x - x_0| < \delta \equiv \min(\delta_1, \delta_2)$. And we are finished.

Theorem: If a function $f = f(x)$ is differentiable at a point c, it is continuous at c. Proof: To show that f is continuous at c we must show that $\lim_{x \to c}[f(x) - f(c)] = 0$, where f is obviously defined at c. Somehow we must use that $\lim_{x \to c}[f(x) - f(c)]/(x - c) = f'(c)$ in our proof. We'll interject it virtually.

$$\lim_{x \to c}[f(x) - f(c)] = \lim_{x \to c}\left[\frac{f(x) - f(c)}{(x - c)} \cdot (x - c)\right] = f'(c) \cdot \lim_{x \to c}(x - c) = 0\,. \tag{26}$$

And we are finished.

Let's go on with something more palatable. Remember integration by parts: $\int udv = uv - \int vdu$? This identity has equivalent forms in differentials and derivatives, but, in contradistinction to convention (what else?), I will not refer to these as integrations by parts, but rather as *contrafluxions*. The distinction is valid since the latter forms are free from the burden of boundary conditions which are needed to deal with the former.

I'm going to be even more belligerent by insisting we view these forms as VEs:

$$du\, v = du\, v + udv - udv = d(uv) - udv\,. \tag{27}$$

We also have that

$$u'v = u'v + uv' - uv' = (uv)' - uv'\,. \tag{28}$$

ed (or transformed) to the

Let's look at an example to illustrate this. Say I'm asked to find the limit $\lim_{u\to 0} \frac{\sin u}{\sqrt{u}}$. I say to myself, "Gee, that looks familiar, like $\lim_{u\to 0} \frac{\sin u}{u} = 1$." (Which, if you didn't already know, you do now.) Then I invoke the VE

$$\lim_{u\to 0} \frac{\sin u}{\sqrt{u}} = \lim_{u\to 0} \frac{\sin\ u}{\sqrt{u}} \cdot \frac{\sqrt{u}}{\sqrt{u}} = \lim_{u\to 0} \frac{\sin u}{u} \lim_{u\to 0} \sqrt{u} = 1 \cdot 0 = 0\,. \tag{29}$$

Another example of RTF is the following (see Einstein [1923], 161): Given that $1/\sqrt{1+\epsilon} \approx 1 - \frac{1}{2}\epsilon$ for $0 < |\epsilon| << 1$, if $\mathrm{g}_{44} \approx 1$, find an approximation to $dx_4 = 1/\sqrt{\mathrm{g}_{44}}$:

$$dx_4 = \frac{1}{\sqrt{\mathrm{g}_{44}}} = \frac{1}{\sqrt{1 + (\mathrm{g}_{44} - 1)}} \approx 1 - \tfrac{1}{2}(\mathrm{g}_{44} - 1)\,. \tag{30}$$

Similar to the last case we have the following (see Frankel (1979, 30–31)): We have the heuristically derived equation for the mass-energy of a fluid spherical blob $B \subset V^3$: $\int_B \rho_0(1 - \frac{1}{2}U)\sqrt{\mathrm{g}_\mathrm{V}}d^3x$ with $\sqrt{-\mathrm{g}_{00}} = 1 - U$:

$$\int_B \rho_0(1 - \tfrac{1}{2}U)\sqrt{\mathrm{g}_\mathrm{V}}d^3x = \int_B \frac{\rho_0(1 - \frac{1}{2}U)}{1 - U}\sqrt{-\mathrm{g}_{00}}\sqrt{\mathrm{g}_\mathrm{V}}\,d^3x \tag{31}$$

$$\approx \int_B \rho_0(1 + \tfrac{1}{2}U)\sqrt{-\mathrm{g}_{00}}\sqrt{\mathrm{g}_\mathrm{V}}\,d^3x\,. \tag{32}$$

Frankel then uses a second VE. For a spherical ball of radius r_0:

$$-\text{ (Newtonian potential energy of } B) = \int_B \frac{\kappa M_r dM_r}{r} = \int_B r\frac{\kappa M_r dM_r}{r^2}$$

where M_r is the mass of a ball of radius r. That this VE is both nontrivial and clever is seen in how it is used. Since the ball is taken to be in equilibrium, the gravitational force is canceled by the hydrostatic force from internal pressure.

$$\int_B r\kappa M_r dM_r r^2 = -\int_B r \cdot 4\pi r^2 dp \tag{33}$$

$$= -4\pi r^3 p\Big|_0^{r_0} + \int_0^{r_0} 3p \cdot 4\pi r^2 dr \tag{34}$$

$$\approx \int_B 3p\sqrt{-\mathrm{g}_{00}}\sqrt{\mathrm{g}_\mathrm{V}}\,d^3x\,. \tag{35}$$

where $p = 0$ at $r = r_0$. Clever!

Why is it that so many important "tricks" and "standard devices" depend on VE? I don't know, really. But I found two beautiful examples in the book *Theory of Distributions: a non-technical introduction* by Richards and Youn

1990, 38). I shall be using quoted passages to let them make their own point without my tainting their testimony. Please excuse that the following is out of context. These two examples concern the definition of "pseudofunctions": "Our objective is to represent the function $1/x^n$ as a distribution. This example is not so special as it might seem: for many other functions with singularities can then be represented by the standard device of separating off poles. For example,

$$\frac{\cos x}{x^4} = \frac{1}{x^4} - \frac{1}{2x^2} + \frac{\cos x - 1 + (x^2/2)}{x^4}\,, \tag{36}$$

the last term being continuous at $x = 0$." I can't help but think that the term "standard device" is vague and fails to inform the reader of the overall pattern it falls into.

And another one:

> **Lemma (The $\varphi(x)/x$ lemma).** Let $\varphi(x)$ be a C^∞ function on R^1 such that $\varphi(0) = 0$. Then the function $\varphi(x)/x$ is C^∞.
>
> **Proof.** Of course, the only difficulty occurs at $x = 0$. We use the following lovely trick. Since $\varphi(0) = 0$,
>
> $$\frac{\varphi(x)}{x} = \frac{\varphi(x) - \varphi(0)}{x} = \int_0^1 \varphi'(xt)\,dt\,. \tag{37}$$
>
> Then, since $\varphi \in C^\infty$, it is clear that we may differentiate under the integral sign as often as we please.Q.E.D. (The authors do not know who first thought of this trick, which we learned from a friend. To appreciate its beauty, try proving the above by any other method.)

The length of an arc in a plane can be calculated with the aid of VE. Let $y = f(x)$ be a function defined on an open continuous subset of the x axis. Let ds be an infinitesimal chord having components dx and dy, then the arc length of the curve between any two points a, b on the interval is

$$\int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \frac{\sqrt{dx^2 + dy^2}}{dx}\,dx = \int_a^b \sqrt{1 + (f'(x))^2}\,dx\,. \tag{38}$$

It was when I solved this next problem that I came to fully appreciate the need to formalize the notion of VE. This is definitely a "feel-good" problem. If you want something somewhere, put it there, but undo the damage to maintain equality. Ok, show that

$$\lim_{h\to 0} (1 + h)^{1/h} = e\,. \tag{39}$$

The question is, Where does the e come from?! Who knows, but we can put it there virtually.

$$\lim_{h\to 0} (1 + h)^{1/h} = \lim_{h\to 0} \exp\left\{\ln(1 + h)^{1/h}\right\} = \exp\left\{\lim_{h\to 0} \frac{1}{h} \ln(1 + h)\right\} = e^1 = e\,. \tag{40}$$

Beautiful. (By the way, $\lim\ e^u = \exp\{\lim u\}$ because the exponential function is analytic.)

Alright, y'all warmed up? What is $\int \sec x\, dx$?

$$\int \sec x\, dx = \int \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x}\, dx \tag{41}$$

$$= \int \frac{d(\tan x + \sec x)}{\tan x + \sec x} \tag{42}$$

$$= \ln|\tan x + \sec x| + c\,. \tag{43}$$

And since you apparently like integrals so much, with $x > 1$:

$$\int \sqrt{\frac{1+x}{1-x}}\, dx = \int \frac{1+x}{\sqrt{1-x^2}}\, dx = \sin^{-1} x - \sqrt{1-x^2} + c\,. \tag{44}$$

This next integration, which is treated as just another ho-hum integration by so many authors, deserves instead to be on a centerfold of some prestigious math journal. It is a perfect example of cogonomics and heuristics, of VE and NBT. *Problem:* Find $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Solution: Let's say we've already tried a bunch of variable substitutions to no avail. What then? (This is where students are most frustrated because they have been systematically denied heuristic and cogonomic training through their education.) The next best thing to try is a *simpler, related problem.* So, we could say to ourselves, Too bad we weren't given the integral $\int_0^{\infty} e^{-x^2} x\, dx$, which is easy to do.

$$\int_0^{\infty} e^{-x^2} x\, dx = -\tfrac{1}{2} \int_0^{\infty} e^{-x^2}\, d(-x^2) = -\tfrac{1}{2} e^{-x^2}\Big|_0^{\infty} = \tfrac{1}{2}\,. \tag{45}$$

But can we use this last integral to help solve our problem? How do we replace the integrand e^{-x^2} with $e^{-x^2}x$? I don't know how to do this in one variable, but I know how to do it in two. Say we have a region in the plane which has rotational symmetry about the origin, then, by changing from rectangular to polar coordinates we have

$$\int\!\!\int f(x,y)\, dx\, dy = \int\!\!\int f(r) r\, dr\, d\theta = 2\pi \int f(r) r\, dr\,. \tag{46}$$

The first step is to note the VE

$$\int_0^{\infty} e^{-x^2} x\, dx = \int_0^{\infty} e^{-r^2} r\, dr\,. \tag{47}$$

Things are looking up. We wanted an r in the integrand, and so we put it there. Now we ask, What $f(x,y)$ will produce $f(r) = e^{-r^2}$? Since $r^2 = x^2 + y^2$ then $f(x,y) = e^{-x^2-y^2}$. Thus $(*)$ becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dx\, dy = 2\pi \int_0^{\infty} e^{-r^2} r\, dr = \pi\,. \tag{48}$$

Now, since $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy$, then by VE,

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx\,dy\,, \tag{49}$$

from which we have that $I = \sqrt{\pi}$. Neat, huh? I just can't shake the feeling that it's magic to solve for an integral on a line by an integral over a plane.

There is a great way to solve integrals of rational functions of $\sin x$ and $\cos x$. The trick is to solve for both trig functions in terms of $\tan \frac{1}{2}x$, then use the substitution $z = \tan \frac{1}{2}x$. Problem: Write $\sin x$ in terms of z. First, we write $\sin x$ in terms of $\tan \frac{1}{2}x$

$$\begin{aligned} \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} &= 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2} \\ &= 2\tan\frac{x}{2}\cdot\frac{1}{\sec^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2z}{1+z^2}\,. \end{aligned} \tag{50}$$

Now we come to a truly amazing theorem. Let $f(x, y, z)$ be a homogeneous nth degree polynomial. Let t be a continuous positive variable. Then

$$f(tx, ty, tz) = t^n f(x, y, z)\,, \tag{51}$$

which is the equation that most authors start with. Next, we just differentiate by t and then set $t = 1$. The variable t rushed in like a whirlwind, and then out it goes again. And when it's finished, this is left

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x, y, z)\,. \tag{52}$$

The derivative of the right side is trivial, so let's look more closely at the left side.

$$\frac{d}{dt}f = \left[\frac{\partial(tx)}{\partial t}\frac{\partial}{\partial(tx)} + \frac{\partial(ty)}{\partial t}\frac{\partial}{\partial(ty)} + \frac{\partial(tz)}{\partial t}\frac{\partial}{\partial(tz)}\right] f \tag{53}$$

$$= \left[x\frac{\partial}{\partial(tx)} + y\frac{\partial}{\partial(ty)} + z\frac{\partial}{\partial(tz)}\right] f = nt^{n-1} f(x, y, z)\,. \tag{54}$$

To finish, all we do is to set t to 1.

Here we come to a few important theorems on linear transformations. They are a little more sophisticated, but worth the effort. For further reference, see Taylor/Mann (1972, 330–333) from which I have drawn heavily. The symbol $||\cdot||$ represents the norm of a vector or linear transformation, as context should make clear.

Theorem 1. If T is a linear transformation from R^n to R^n (called an *endomorphism* of R^n) and $||\,T\,|| < 1$, then $I - T$ is invertible, and $||\,(I-T)^{-1}\,|| \leq 1/(1 - ||\,T\,||)$.

Proof: We will show that $I - T$ is invertible because it maps nonzero vectors to nonzero vectors. With the assumption that the norm of a nonzero vector is nonzero, we have

$$||\,(I-T)\mathbf{x}\,|| \geq \big|\,||\,I\mathbf{x}\,|| - ||\,T\mathbf{x}\,||\,\big|\,, \tag{55}$$

with

$$||\,I\mathbf{x}\,|| = ||\,\mathbf{x}\,|| \quad \text{and} \quad ||\,T\mathbf{x}\,|| \leq ||\,T\,||\,||\,\mathbf{x}\,||\,. \tag{56}$$

Thus,

$$||\,(I-T)^{-1}\mathbf{x}\,|| \geq (1 - ||\,T\,||)||\,\mathbf{x}\,||\,, \tag{57}$$

where $1 - ||\,T\,|| > 0$ from the initial stipulation. Thus $I - T$ is invertible.

Now for part 2 of this proof. Since $I - T$ is invertible for all $\mathbf{y}$ in R^n

$$\mathbf{y} = (I-T)[(I-T)^{-1}\mathbf{y}] \quad \text{(by virtual emplacement)}$$

On replacing $\mathbf{x}$ by $(I-T)^{-1}\mathbf{y}$ in (57), we get

$$||\,\mathbf{y}\,|| = ||\,(I-T)[(I-T)^{-1}\mathbf{y}]\,|| \geq (1 - ||\,T\,||)||\,(I-T)^{-1}\mathbf{y}\,||\,. \tag{58}$$

Thus, for all $\mathbf{y}$ in R^n

$$||\,(I-T)^{-1}\mathbf{y}\,|| \leq \frac{||\,\mathbf{y}\,||}{1 - ||\,T\,||}\,, \tag{59}$$

implying that $||\,(I-T)^{-1}\,|| \leq 1/(1 - ||\,T\,||)$.

Corollary: If $||\,I - T\,|| < 1$ then T is invertible.

Proof: Let $K = I - T$. From the theorem, if $||\,K\,|| < 1$, then $I - K$ is invertible. Now, assuming $||\,K\,|| < 1$ then $||\,I - T\,|| < 1$, which implies that $I - K$ is invertible, which also implies that $I - (I - T)$ is invertible. But $I - (I - T) = T$, hence, T is invertible.

Theorem 2 The set Ω of invertible linear operators is an open subset of End R^n.

Proof: A set is open if every point of the set is an interior point. Now, let $T \in \Omega$, then for any $L \in \Omega$

$$L = T - (T - L) = T[I - T^{-1}(T - L)]\,. \tag{60}$$

But $||\,T^{-1}(T-L)\,|| \leq ||\,T^{-1}\,||\,||\,T - L\,||$. Now, requiring that $||\,T - L\,|| < 1/||\,T^{-1}\,||$, we have that $||\,T^{-1}(T-L)\,|| < 1$, thus, by Theorem 1, $I - T^{-1}(T-L)$ is invertible. Thus,

$$L = T[I - T^{-1}(T - L)] \tag{61}$$

is also invertible. So, T is at the center of an open ball of radius $1/||\,T^{-1}\,||$, which is wholly contained in Ω.

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