Virtual Emplacement 1993

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Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, July, 1993, under the title of *How did you know to do that?* It expands the examples of virtual emplacement from previous articles. Note: The part of this paper that dealt with stuff from Donald Knuth is left out for the time being, until I can perfect its presentation.

Typically, students are taught that all they need do to be a good problem solver is to learn how to apply book methods to certain classes of problem. Those students who haven't become completely discouraged or intimidated, usually get around to asking the teacher after a trivial but nonintuitive step is presented: "How did you know to do that?" A good answer rarely given, for to do so would require the teacher to understand the formal role of *cogonomics* in teaching.

Cogonomics is a term I coined to go hand-in-hand witrh the term *didactic*. The latter term means the scientific study on how to teach. The former means the scientific study on how humans are efficient at learning and understanding. Cogonomics is the ergonomics of the mind, of human understanding.

Before going on, we need to have some definitions ready for use: A *mathematical expression* (ME) is a combination of numbers and symbols without any *relational operators* (RO), which are any of the objects $\{<, >, \leq, =, \geq\}$. A *mathematical sentence* (MS) is the assertion that two expressions (namely, that LHS and RHS of a relation) are actually related by the relational operator that connects them.

The present paper is a return to the concept of virtual emplacement (VE), by which one can:

I) change the form of an ME without changing its value

II) change the form of an MS without changing the statement's RO

III) change the form of an MS with one RO to an MS with another RO

IV) employ a step in a proof that is either a do/undo operation or else introduce something that seems to have more content than the steps prior to it can justify

Let's look at some examples. *Identities* are born from Type I VEs, such as $a = a + b - b = (a + b) - b$. Representative of a Type II VE is: if $a = b$ then $a + c = b + c$. For Type III VEs we have: c is a positive number and $a + c = b$ iff $a < b$ (or in other words a "is less than" b).

We begin our examples with a Type III VE from operations research, which is concerned with optimizing use of limited or expensive resources. Often in this research one encounters constraints in the form: $\sum_{j=1}^{n} a_j$ RO b with $b \geq 0$, where the preferred RO is equality, but is given in some other form. For example, the inequality $x_1 + 3x_2 + 117x_3 \leq 500$ becomes $x_1 + 3x_2 + 117x_3 + x_4 = 500$. We call x_4 a "slack" variable. The interested student can pursue this further on his or her own from here. (The conversion of inequaltities to equalities by using slack variables is the Next Best Thing to having equalities to start with.)

Our next example comes from the study of *continued fractions* (CFs). In computer computations with transcendental reals, such as π, e, and $\sqrt{2}$, we need to find an optimal way to represent approximations of these numbers by fractions. Now we can just take the number's decimal representation and truncate it at whatever decimal point we want, but there is a better way.

Using virtual emplacement, recursion, simple algebra we can produce a CF representation as follows: Take the transcendental as a sum of its integer and decimal parts. Then repeatedly "rewrite the current decimal part as unity over the decimal's multiplicative inverse, producing a denominator with a number having both integer and decimal parts itself" until you have the desired accuracy of representation.

Let's try this out on the number π.

$$\begin{aligned}
3.14159265\ldots &= 3 + .14159265\ldots \\
&= 3 + \frac{1}{(.14159265\ldots)^{-1}} \\
&= 3 + \frac{1}{7.062513305\ldots} \\
&= 3 + \frac{1}{7 + \frac{1}{(.062513305\ldots)^{-1}}} \\
&= 3 + \frac{1}{7 + \frac{1}{15 + .9965944\ldots}} \\
&\approx 3 + \frac{1}{7 + \frac{1}{16}} \\
&= \frac{355}{113}\,.
\end{aligned}$$

Thus, every real number can be given the alternative representation as a CF by

$$r = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}$$

or, alternatively, as the series $\{a_0, a_1, a_2, \ldots\}$.

Now we take an example from simple algebra. When students are taught how to divide one polynomial by another, they usually have no problem using the algorithm for something like $(x^2+3x+2)/(x+1)$. They simply start with

$$x+1\,\overline{\big)\,x^2+3x+2}$$

and proceed easily. But give them something like $(x^2+2)/(x+1)$ and they can get confused. Telling the student to put in the linear term virtually (meaning, with coefficient zero) to start with

$$x+1\,\overline{\big)\,x^2+0x+2}$$

is fine as far as it goes, but algebra is the ideal place to begin the systematic teaching of the concept of virtual emplacement and this is a good example with which to emphasize it.

And now, just for fun, we introduce complexity built out of absolutely nothing:

$$0=(1-1)^{2k+1}=\sum_{i=0}^{2k+1}\binom{2k+1}{i}(-1)^i\,.$$

We come now to the fourth and last Type of VE defined so far—that of putting steps in a logical proof. This analysis is directly applicable to symbolic logic, to argumentation, and to the reasoning behind mathematical proof. However, I must of necessity assume that the reader has some knowledge of symbolic logic. Consider the following "proof" (it's not a real proof of anything) to demonstrate Type IV VEs:

1	$\mid$ A	Given
2	$\mid$ $A \wedge (B \vee \sim B)$	Conjunction with tautology
3	$\mid$ $(A \wedge B) \vee (A \wedge \sim B)$	Distributive
4	$\mid$ C	Given
5	$\mid$ $C \vee D$	Addition
6	$\mid$ $E \wedge F$	Given
7	$\mid$ E	Simplification
8	$\mid$ $G \vee H$	Given
9	$\mid$ $\sim\sim(G \vee H)$	Double Negation
10	$\mid$ $\sim(\sim G \wedge \sim H)$	DeMorgan's Theorem

Steps 1–3, 4–5, 6–7, and 8–10 are four groups that are each independent of the others. I dare not say that these are the only VEs in symbolic logic, but I can say that the conceptual difficulty of virtual emplacement has not escaped the attention of prominent authors. One such is Howard Kahane[1] who tells of

[1]H.Kahane. 1969 [1973]. *Logic and Philosophy*, 2ed. Wadsworth Publishing Co., Inc. Belmont, California.

how difficult it is to get students to use *Addition* even after he proves it's a valid technique (p. 57): "Even after proving that Addition is a valid argument form, students are reluctant to use it, because they believe that somehow it is 'cheating' to be able to 'add' letters to a line simply at will." But this is, afterall, part of the great fascination and wonder of mathematics. Long before I read this text, I discovered the rule of virtual emplacement *If you want something somewhere, stick it there*—and then undo the effect of what you did.

Now we turn to combinatorics. It's a well-known binomial identity that

$$\binom{n}{k} = \binom{n}{n-k} \tag{1}$$

The logic of the proof is simple: We will start off in n, k-space (where the only variables we're allowed to use are n and k) and then transform into n, m-space (where the only variables we're allowed to use are n and m. Once in the new space, we perform some identity operations, and then transform back into n, k-space, leaving an expression that is equal to the original expression. See the figure below.