Window shopping for algebras

P. Reany

February 19, 2020

Abstract

This paper is a partial reprint of an article that first appeared in the *Arizona Journal of Natural Philosophy*, April, 1992. The phenomenal growth of mathematical physics over the past hundred years has brought us today to the task of having to learn from a plethora of algebras and then to choose the appropriate algebra to fit a particular problem or subject. Not all algebras are equally facilitating to problem solving. This essay promotes the use of those algebras that maximize the use of time-honored problem-solving techniques, especially the method of virtual emplacement.

1 Introduction

In the October 1991 issue I wrote about how Einstein viewed the placement of coordinates as a virtual emplacement into the affairs of the universe. They're there in the formalism, but not there in reality. Einstein decided that the best way to remove any spurious effect of introducing coordinates into theory is to obviate such an effect by demanding that the laws of physics should be covariant with respect to coordinate transformations. That way they're not there even though they are.

Einstein chose as his preferred mathematical formalism tensor analysis. As a result he had to deal with drawbacks of it which aren't present in other systems, such as differential forms or geometric calculus. Today, we are a long way from having to settle for the formalism that so and so say we should use just because that's the way it was always done. We can now choose from a very wide variety of formalisms. It's true that it takes time to learn a number of formalisms to be able to choose intelligently, but nobody ever claimed that mathematical physics is going to be easy. A very good place to start is with an introduction to abstract algebra.

In this paper I will argue that a methodology of problem solving, which I call the *method of virtual emplacement* or MVE, is a very powerful technique that takes on many forms in the various systems. The technique itself is not difficult in any form. In fact, it looks so deceptively simple that my critics are sure to have a good laugh at it. But my emphasis in covering it is to promote the art of problem solving. The trick isn't in applying a VE technically correct; the trick is in knowing which form of VE if any should be used at which point—that's the object of my presentation.

I intend to show that VE is so powerful that all effort should be made to choose those formalisms that maximize the number of VE types for use. This will be accomplished in two steps. The first is to give an introduction to some of the algebras most useful in mathematical physics, and the second is to present a few examples as illustration. However, most of the examples—from arithmetic to integrals of the theory of gravitational curvature—will be given in a later issue of the journal.

Since this is an essay and not a dissertation on all types of algebras, I will present only the most important to mathematical physics and emphasize their use to problem solving.

Basically, an algebra is a set of elements which is closed under one or more operations. For example, the integers are closed under addition. This example is also an example of the first class of algebras I'll discuss—the *group*, which is defined as follows. G is a group if

1) G is a set of elements under a binary operation $*$ such that for all $g, h \in G$

$$g * h \in G$$

2) G is associative, meaning that for all $a, b, c \in G$

$$(a * b) * c = a * (b * c)$$

3) there exists a unique element e such that for all $g \in G$

$$e * g = g * e = g$$

4) for all $g \in G,\ g^{-1} \in G$ such that

$$g * g^{-1} = g^{-1} * g = e$$

A group can be finite or infinite. It can be abelian (commutative) or nonabelian (noncommutative).

Next we'll study the concept of a ring. A *ring* R is a set of elements with two binary operations $+, *$. The first operation, which we'll call "addition," forms an abelian group $(R, +)$. The second operation $*$, which may or may not be commutative, must distribute over $+$. Thus, for all $a, b, c \in R$

$$a * (b + c) = a * b + a * c\,, \tag{1}$$

which is left-sided distribution; the multiplication $*$ must also distribute from the right.

One interesting property of some rings is that of the zero-divisor. A *zero-divisor* is a nonzero element which when it multiplies some other nonzero element yields zero.

Our next algebra is the *field*, which is a commutative ring with a multiplicative identity (unity) and every nonzero element has an inverse. So if you're doing 2nd-order linear differential equations or the theory of algebraic equations, then it seems that all you need is the field of complex numbers. Well, there's a big difference between that which is just sufficient and that which is best. And when there's a choice to be made only you can decide which is best for you, your needs and temperment.

For my money, I'll take the ring $\mathcal{C}_\infty$ (Clifford 1) over the field of complex numbers every time. The reason $\mathcal{C}_\infty$ is better is that it supports more tricks, such as VE, than does the complex field. I offer as examples the three articles in this issue.

Our next algebra is that of the vector algebra. Between mathematics usage and physics usage there is a disagreement about how to describe vector algebras; the details of which I won't go into at this point.

We can define a vector space $\mathcal{V}$ as a set of elements that form a group under a commutative binary operation $+$. Furthermore, we allow for associative scalar multiplication, where the scalars are from some field $\mathcal{F}$, and which satisfy the additional properties for all $x, y \in \mathcal{V}$:

1) $1x = x$
2) for all scalars α: $\alpha(x + y) = \alpha x + \alpha y$
3) for all scalars α, β: $(\alpha + \beta)x = \alpha x + \beta x$
4) for every scalar α: $\alpha x = x\alpha$

By our definition of an algebra a vector space is also an algebra; however, most physicists view a vector algebra as a vector space extended to include at least one "vector product." A vector product is a binary operation other than addition that distributes over vector addition. A familiar example of this is the Gibbs's vector algebra with its cross and dot products. The cross product maps two vectors in the space to some vector in the space, but the dot product doesn't, since it maps two vectors to a scalar. This inconsistency is resolved differently by three main camps.

The first camp, that of most physicists, is simply to ignore the discepency altogether. The second camp, that of most mathematicians is to define a space dual to the original space, and define a mapping between the vector space and its dual to the field of scalars. The third camp, that of those adopting the way of Clifford, is to form an algebra that contains vectors, multivectors, and scalars. In doing so, the inner, or dot, product of vectors is in the same vector space. Furthermore, the dot and vector products defined on vectors of a Clifford algebra can be expressed in terms of an associative "geometric" product. We shall see that many of the most powerful and elegant VEs depend on an associative product.

Definition: The *method of virtual emplacement* is the transformation of the form of a mathematical expression while leaving its value unchanged.

I can just hear the polemics already: “That’s just using identities; there’s nothing to this nonsense!” I will give my critics even more to debunk by starting off my examples from arithmetic. The purpose of studying a multitude of examples is to develop a knack for intuiting where and when a virtual emplacement could or *should* be used. In truth, their use is not always obvious. We’re talking cogonomics here.

Problem 1: Let a, b stand for any two particular integers. By the method of VE show that $a - b = -(-a + b)$.

$$a - b = [(-1)(-1)]a + (-1)b \tag{2}$$
$$= (-1)[(-1)a] + (-1)b = (-1)[(-1)a + b] \tag{3}$$
$$= -(-a + b)\,. \tag{4}$$

We have set the problem in the realm of the ring of integers. The VE was, of course, $a = 1 \cdot a = (-1)(-1)a$, but by itself that’s not enough. The second part of the magic is the fact the multiplication is associative, thus in the next step the negatives were re-associated. Note also that the distributive property will be nearly as useful as associativity for VE.

Problem 2: If a, b, c are three real numbers such that $a < b < c$, show that $c - a = (c - b) + (b - a)$.

$$c - a = c + 0 - a \tag{5}$$
$$= c + (-b + b) - a \tag{6}$$
$$= (c - b) + (b - a)\,. \tag{7}$$

No explanation is needed. I included it to complement the previous VE, which was based on the fact that in any ring with unity we can multiply any term by unity. This VE is based on the fact that in any group with an additive identity, 0, that zero can be added to any expression without changing it.

A similar VE can be found on page 10 of the January 1992 issue:

$$\frac{\partial F}{\partial \eta_e}\delta\eta_e = \frac{\partial F}{\partial \eta}\delta\eta\,. \tag{8}$$

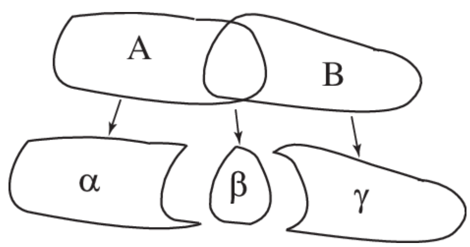
In a suitable integration we can combine both types of VEs. Show that, if a, b are constants

$$\int \frac{dx}{ax + b} = \frac{1}{a}\log(ax + b) + c\,. \tag{9}$$

Proof:

$$\begin{aligned}\int \frac{dx}{ax+b} = \frac{a}{a}\int \frac{dx}{ax+b} &= \frac{1}{a}\int \frac{d(ax)}{ax+b} \\ &= \frac{1}{a}\int \frac{d(ax)+db}{ax+b} \\ &= \frac{1}{a}\int \frac{d(ax+b)}{ax+b} \\ &= \frac{1}{a}\log(ax+b)+c\,.\end{aligned}$$

And now for another act of legerdemain, consider the cardinality of sets. Find an equivalent expression for the cardinality of $A \cup B$. I've left the problem open-ended to allow me to illustrate an important but tangential principle of mathematical aesthetics. Let's represent the cardinality of a set X by $m(X)$. Referring to the creative redrawing of the standard figure (both displayed right), we have that



$$\begin{aligned} m(A\cup B) &= m(\alpha)+m(\beta)+m(\gamma) &(10)\\ &= m(\alpha)+m(\beta)+m(\gamma)+[m(\beta)-m(\beta)] &(11)\\ &= [m(\alpha)+m(\beta)]+[m(\gamma)+m(\beta)]-m(\beta) &(12)\\ &= m(A)+m(B)-m(A\cap B)\,. &(13)\end{aligned}$$

We can stop at this point because 1) the result is in terms of A and B, and 2) the result is in a form symmetric in A and B just as $m(A \cup B)$ is.

Multiplying by unity and adding in zero aren't the only VEs. Another powerful one is the ability to choose the real or imaginary part of a complex number. For example,

$$\cos\theta = \mathrm{Re}[\cos\theta] = \mathrm{Re}[\cos\theta + i\sin\theta] = \mathrm{Re}[e^{i\theta}]\,. \tag{14}$$

As an example, consider solving for $\cos 2\theta$ in terms of $\cos\theta$:

$$\begin{aligned}\cos 2\theta = \mathrm{Re}[e^{2i\theta}] &= \mathrm{Re}[(e^{i\theta})^2] &(15)\\ &= \mathrm{Re}[(\cos\theta + i\sin\theta)^2] &(16)\\ &= \cos^2\theta - \sin^2\theta\,. &(17)\end{aligned}$$

In Clifford algebras one can select for more than just real or imaginary parts. One can also select for scalar, vector, etc parts.

Each of us is a measure unto ourselves about how we wish to bring mathematics and physics into our personal realm of knowledge. There is no absolute standard. The past is no guarantee of the best path to take. We have to study for oursleves, then choose for ourselves. That's the privilege and burden of having a choice.