Area of a Parallelogram

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Abstract

We show here how to relate the area of a parallelogram in the usual manner to how it can be represented in **geometric algebra**.

Our basic goal here is to represent the area of a parallelgram in geometric algebra, connecting it up with its standard presentation in geometry.

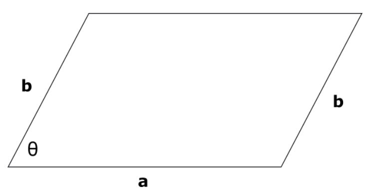

Figure 1. We have here a typical parallelogram.

In Fig. 1 we see a parallelogram whose area we know from high school math to be

$$\text{Area Parallelogram} = ab\sin\theta\,, \tag{1}$$

where θ is the angle between sides $\mathbf{a}$ and $\mathbf{b}$, and a and b are, respectively, the lengths of sides $\mathbf{a}$ and $\mathbf{b}$.

Let's now place vectors on the sides of the parallelogram so that we can represent its area with a bivector (which better than a cross product).

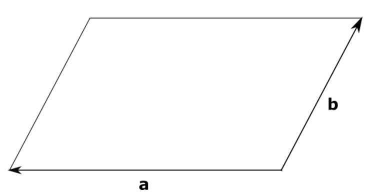

Figure 2. Written in bivector form, the area of the parallelogram can be expressed as $\mathbf{a} \wedge \mathbf{b}$.

Now, let $\mathbf{B}$ be any nonzero bivector represented as the wedge product of any two vectors (such as in Fig. 2). In geometric algebra, the *magnitude* of this bivector is defined by

$$|\,\mathbf{B}\,| = [\langle \mathbf{B}^\dagger \mathbf{B} \rangle]^{1/2}\,, \tag{2}$$

where $\langle \cdots \rangle = \langle \cdots \rangle_0$ means to take the scalar part of what's between the brackets. The symbol † is the *reverse* operator, which means to take the ordering of the vectors in reverse order. This operation distributes over addition. Scalars and vectors are invariant under the reverse operation. For clarification, with the a_i's as vectors, we have the general relations:

$$\begin{aligned}(a_1 a_2 \cdots a_n)^\dagger &= a_n \cdots a_2 a_1\,, \\ (a_1 \wedge a_2 \wedge \cdots \wedge a_n)^\dagger &= a_n \wedge \cdots \wedge a_2 \wedge a_1 \\ (a_1 \wedge a_2)^\dagger &= -a_1 \wedge a_2\,.\end{aligned} \tag{3}$$

Some additional background in geometric algebra may be of help. The *geometric product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given as

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}\,. \tag{4}$$

Solving for the bivector part, we have

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{ab} - \mathbf{a} \cdot \mathbf{b}\,. \tag{5}$$

We also need to know that

$$\mathbf{a} \cdot \mathbf{b} = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba})\,, \tag{6}$$

and, of course, the familiar relation from vector algebra,

$$\mathbf{a} \cdot \mathbf{b} = ab \cos\,\theta\,. \tag{7}$$

Now, for any two multivectors A and B

$$\langle AB \rangle^\dagger = \langle B^\dagger A^\dagger \rangle\,. \tag{8}$$

As a corollary, if A and B are both bivectors, then

$$\langle AB \rangle^\dagger = \langle (-B)(-A) \rangle = \langle BA \rangle\,. \tag{9}$$

My first task is to show that $|\,\mathbf{B}\,|$ is the area of the bivector parallelogram, and is given as $ab \sin\theta$. For convenience, I'll start with $|\,\mathbf{B}\,|^2$ where $\mathbf{B} = \mathbf{a} \wedge \mathbf{b}$.

$$\begin{aligned}|\,\mathbf{B}\,|^2 &= \langle \mathbf{B}^\dagger \mathbf{B} \rangle \\ &= \langle (\mathbf{a} \wedge \mathbf{b})^\dagger \mathbf{a} \wedge \mathbf{b} \rangle \\ &= \langle (\mathbf{ab} - \mathbf{a} \cdot \mathbf{b})^\dagger (\mathbf{ab} - \mathbf{a} \cdot \mathbf{b}) \rangle \\ &= \langle (\mathbf{ba} - \mathbf{a} \cdot \mathbf{b})(\mathbf{ab} - \mathbf{a} \cdot \mathbf{b}) \rangle \\ &= \langle \mathbf{baab} - \mathbf{a} \cdot \mathbf{b}(\mathbf{ba} + \mathbf{ab}) + (\mathbf{a} \cdot \mathbf{b})^2 \rangle \\ &= \langle \mathbf{a}^2 \mathbf{b}^2 - 2(\mathbf{a} \cdot \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 \rangle \\ &= a^2 b^2 - a^2 b^2 \cos^2\theta \\ &= a^2 b^2 \sin^2\theta\,.\end{aligned} \tag{10}$$

On taking square roots of both sides, this result agrees with what we knew from (1). Now, a useful corollary: if A is a scalar or a vector, then

$$A^\dagger = A\,. \tag{11}$$

My last task is to show that

$$|\,\mathbf{B}\,| = |\,\mathbf{B}\cdot\boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2\,|\,, \tag{12}$$

where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are unit vectors along the x- and y-axes, respectively.

Next follows some useful prerequisite results: Let A and B be any two multivectors. If $\mathbf{h}$ is any multivector such that $\mathbf{h}\mathbf{h}^\dagger = 1$, then

$$AB = A(1)B = A\mathbf{h}\mathbf{h}^\dagger B = (A\mathbf{h})(\mathbf{h}^\dagger B)\,, \tag{13}$$

which comes from the fact that the geometric product is associative! (Yay!) In particular, let $\mathbf{h} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, then $\mathbf{h}\mathbf{h}^\dagger = 1$. (Prove it! Hint: First, $\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = 1$, and, second, $\mathbf{h}\mathbf{h}^\dagger = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$, and then employ associativity carefully and remember that scalars commute with any multivector.) So,

$$AB = A(1)B = (A\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)(\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 B)\,. \tag{14}$$

Now,

$$\begin{aligned}
|\,\mathbf{B}\,|^2 &= \langle \mathbf{B}^\dagger\mathbf{B}\rangle \\
&= \langle \mathbf{B}^\dagger\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\mathbf{B}\rangle \\
&= \langle (\mathbf{B}^\dagger\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)(\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\mathbf{B})\rangle \\
&= \langle (\mathbf{B}^\dagger\cdot\boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2)(\boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1\cdot\mathbf{B})\rangle \\
&= (\mathbf{B}^\dagger\cdot\boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2)\langle (\boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1\cdot\mathbf{B})\rangle \\
&= (\mathbf{B}^\dagger\cdot\boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2)\langle (\boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1\cdot\mathbf{B})\rangle^\dagger \quad \text{from (11)} \\
&= (\mathbf{B}^\dagger\cdot\boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2)^2 \quad \text{from (8)} \\
&= |\,\mathbf{B}^\dagger\cdot\boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2\,|^2 \\
&= |\,\mathbf{B}\cdot\boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2\,|^2\,.
\end{aligned} \tag{15}$$

We get (12) by taking square roots of both sides. QED

Additional note. The reader who is not familiar with geometric algebra and wedge products, can reference on-line sources through the search string ‘wedge product’. The main reference book is *New Foundations for Classical Mechanics* by David Hestenes (Kluwer Academic Publishers). An online reference is

http://geocalc.clas.asu.edu/pdf-preAdobe8/PrimerGeometricAlgebra.pdf