The Logic of an 'if and only' Proof

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I'm writing this short note as a reminder to myself what is meant by an 'if and only' (iff) proof. When I read a theorem that claims

$$A \text{ iff } B \,, \tag{1}$$

where A and B are propositions, I always have to figure out from first principles what this seemingly simple statement means. Well, what does the 'iff' in (1) mean? It means the conjunction of two propositions at once, which is,

$$A \text{ if } B \quad \text{and} \quad A \text{ only if } B \,. \tag{2}$$

Well, 'A if B' (or rather, 'if B then A') means that A is true if B is true. Or, expressed another way, B implies A. Expressed one more way

$$A \text{ if } B \quad \text{means} \quad A \Longleftarrow B \,, \tag{3}$$

which respects the order in which propositions A and B are presented. So to prove this half of the conjunction, we can symbolized it by ($\Longleftarrow$).

Now, on to the second half of the conjunction: 'A only if B' means that A cannot be true unless B is true. Or, expressed another way,

$$\text{not } B \Longrightarrow \text{not } A \,. \tag{4}$$

But this is equivalent to its contrapositive

$$A \Longrightarrow B \,, \tag{5}$$

which is symbolized as ($\Longrightarrow$).

So, if you want to assume A is true and then prove B, use the right-pointing arrow. Otherwise, use the left-pointing arrow.