Honk if you hate mathematical induction

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Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, July, 1991. Russell and Poincare, taking widely differing views on mathematical induction, argued about what it meant and how it was to be justified. Poincare thought of it as mysterious; Russell claimed to have removed its mystery. Decades later, today's student must wrestle with it to solve textbook problems, but how often do they understand what they're doing?

1 Introduction

Today's investigation is on the mysteries of mathematical induction (MI). It's not that I have trouble believing in MI; it's that I have always had trouble following the logic of the proofs presented to me. It makes me feel a little less insecure to note that Russell and Poincare couldn't agree on its meaning. In his book *Introduction to Mathematical Philosophy* (1919,27), Russell argues that the prior association of mystery had been removed by the logical foundation of mathematics. I have to admit still feeling a sense of awe in MI. (Russell was always trying to take the fun out of mathematics. Ultimately, however, thanks to our hero, Gödel, he failed.) See Kline (1980,233) for some history on the disputations between well-known mathematicians about the meaning of MI.

The first time I took a serious look to understand the principle of mathematical induction was in abstract algebra. My text was *Abstract Algebra: A First Course* by Saracino (1980). On page 4 of the text appears the theorem and its proof. I will paraphrase the theorem for you but I won't present his proof (note: we'll take the base case to be unity for this discussion).

Theorem on Mathematical Induction: Suppose that $P(n)$ is a propositional function that maps the positive integers $\mathbb{Z}^+$ to true or false (such a function is said to be *boolean-valued*). If

I) $P(1)$ is true (established by actually demonstrating it), and

II) for some $k > 1$, the assumption that $P(k)$ is true then implies that $P(k+1)$ is true then $P(n)$ is true, meaning that $P(k)$ is true for all $k \in \mathbb{Z}^+$.

The act of assuming that $P(k)$ is true has a name: ‘applying the *inductive hypothesis*’.

The part I particularly had trouble with in application is (II): How do I apply it to particular problems? The reason that I can understand condition (II) at this time is because I've become accustomed to the weird nature of material implication (having studied it for the last few years) that is used in it.

Now I want to present to you another contender for the principle of MI. You decide if they're really the same thing. From *A First Undergraduate Course in Abstract Algebra* (2nd ed) by Hillman & Alexanderson (1978) page 4, we read:

Axiom 1: The Well Ordering Principle Every nonempty subset S of $\mathbb{Z}^+$ has a least element.

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Theorem 1 Principle of Mathematical Induction

Let S be a set of positive integers, that is, a subset of $\mathbb{Z}^+$, with the properties:

(a) *The positive integer 1 is in S,*

(b) *If a positive integer k is in S, so is $k + 1$.*

Then every positive integer is in S; that is, $S = \mathbb{Z}^+$.

I will first prove this version of MI, which I refer to as *MI for integers*, then I will prove Saracino's version, which I refer to as *MI for propositions*, as a corollary to the the former. I have included the Well-Ordering Principle because I need it in the proof. The Well-Ordering Principle claims that every nonempty subset of the positive integers contains a least element.

Proof:

From (a) we know that S has at least one positive integer, namely 1. Let's define T as the set of all positive integers not in S. Now, it is certainly true that either T is empty or it is not. If it is empty then we are finished because $S = \mathbb{Z}^+$. If it is not empty, then, by the Well-Ordering Principle, it has a least element, which we can identify without loss of generality as $t + 1$.

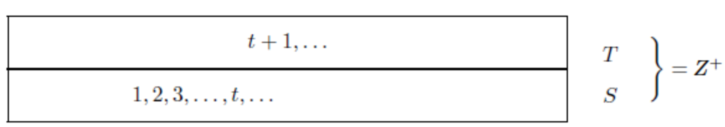

Figure 1. t is in S, but t + 1 isn't.

Thus, as is readily seen from Figure 1, S has the element t but not the element $t+1$, which contradicts condition (b). Therefore T is empty and we are finished.

It's important to note, however, that the two conditions (a) and (b) need not be true for a set S: What is the case is that the theorem *is* true *if* conditions (a) and (b) *are* true.

Now we can prove the propositional form of MI as a corollary to this theorem. Connecting the two is simple. We just define the set S by the kth proposition: $S = \{k \in \mathbb{Z}^+ : P(k)\}$. Now, if $P(1)$ is true (call this Condition 1) then condition (a) of the theorem is true. And if it is true that *if* $P(k)$ *is true (call this Condition 2) then so is* $P(k+1)$, then condition (b) is true, too. Therefore, by the theorem, $S = \mathbb{Z}^+$, so that $P(k)$ is true for all $k \in \mathbb{Z}^+$, or in other words $P(n)$ is true. The corollary is proved.

It's good to see this proof in the following figure.

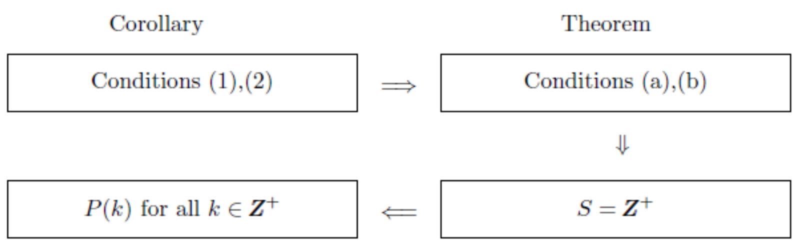

Figure2.

Now we can use this last figure as an exemplar for the study of the relation between a theorem and its corollary by examining the following figure.

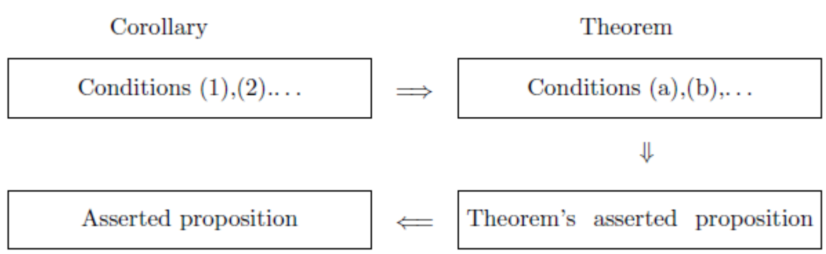

Figure 3.

Now let's go back to reconsider the meaning of condition (2) of the propositional form of MI. Saracino wrote: (2) for every positive k, if $P(k)$ is true, then $P(k+1)$ is true. The meaning of which, though subtle, can be made clear by interpreting (2) in terms of (b) in the integer form of MI. From (b) we have "for every positive integer k, if $P(k)$ is true, then $P(k+1)$ is true" means that for every positive integer k, if $k \in S$ then $k+1 \in S$." Another way to put (b) is "whenever $k \in S$ then $k+1 \in S$," which does not mean that $k+1 \in S$ *because* $k \in S$.

This next statement may seem outlandish, but the following form of MI in symbolic logic

$$[P(1) \cap (\text{for all } k \in \mathbb{Z}^+(P(k) \supset P(k+1))] \supset P(n)$$

is easier to understand because it defines itself in terms of the well-defined material implication. For further information see Kline (1980,187-9) or any text on symbolic logic.

Hillman, A.P. and Alexanderson, G.L. 1978. *A First Course in Abstract Algebra* 2nd ed. Belmont, California: Wadsforth Publishing Company.

Kline, M. 1980. *Mathematics, the Loss of Certainty.* New York: Oxford University Press.

Russell, B. [1919]. *Introduction to Mathematical Philosophy.* New York: Simon and Schuster.

Saracino D. 1980. *Abstract Algebra: A First Course.* Massachusetts: Addison-Wesley Publishing Company.