Mathematical Induction for Students

P. Reany

February 19, 2020

Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, April, 1993, which was itself a reprint from the January 1987 issue. If you've ever wished for a better way to use mathematical induction in proofs, here it is, though this techinque works best on problems that reduce finite sums or products to closed forms.

1 Introduction

Our purpose is to identify these common steps and to use them to prove theorems applicable to two classes of induction problems: the reduction of finite sums and finite products to closed forms.

2 Abstract and applied mathematical induction

Given two functions $L(n)$ and $R(n)$, whose domains are the integers greater than or equal to some integer m, that is, the base case, which ios often zero or one, if we propose that

$$L(n) = R(n)\,, \tag{1}$$

where "$L(n)$" is the left expression and "$R(n)$" is the right expression," we can, in principle, test the validity of (1) for a finite set of integers starting at the smallest integer m; but if we propose that (1) is true for all integers n such that $n \geq m$, we must often rely on the principle of mathematical induction. A standard approach to mathematical induction is the propositional approach described below.[1]

Let $P(n)$ be the proposition that the algebraic sentence (1) is true for all integers n such that $n \geq m$. Let $P(m)$ be the validity of the proposition that $L(m) = R(m)$, similarly, let $P(k)$ be the validity of $L(k) = R(k)$. Then, to prove that $P(n)$ is true by mathematical induction, we must

1) show that $P(m)$ is true
2) assume $P(k)$ is true for some arbitrary $k \geq m$

3) show, on the basis of 2), that $P(k+1)$ is true

We can now associate an algebraic interpretation of the above steps. As before, $P(m)$ is true if and only if $L(m) = R(m)$. Assuming that $P(k)$ is true is logically equivalent to assuming that $L(k) = R(k)$. Then, to show that $P(k+1)$ is true we need to show that

$$L(k+1) = R(k+1) \tag{2}$$

given that $L(k) = R(k)$.

Up to this point we have been quite general, but now we consider the special class of problems where $L(n)$ is a finite sum and $R(n)$ is a closed form. Let

$$L(n) = a_m + a_{m+1} + a_{m+2} + \cdots + a_n\,, \tag{3}$$

and let $R(n)$ be any closed form such that $P(n)$ is the proposition that $L(n) = R(n)$ for all $n \geq m$. Considering only those cases where $P(m)$ is true, all we need do is to prove $P(n)$ is to assume $P(k)$ to prove $P(k+1)$

According to our procedure, we assume $L(k) = R(k)$, then $L(k+1) = R(k+1)$ if and only if

$$R(k+1) = R(k) + [L(k+1) - L(k)] = R(k) + a_{k+1}\,,$$

in which case then $P(k+1)$ is true. We have just proved the following theorem:

THEOREM 1 If $P(n)$ is the proposition that $L(n) = R(n)$ for all integers $n \geq m$, we bgin by demonstrating that the proposition is true for the base case: $L(m) = R(m)$. Then, if $L(n)$ is given by (3) and if $P(k)$ being assumed true for any $k \geq m$ demands that $P(k+1)$ also be true on the same domain, then to prove $P(n)$ it is sufficient to prove that

$$R(k+1) = R(k) + a_{k+1}\,, \tag{4}$$

or, equivalently, that

$$a_{k+1} = R(k+1) - R(k)\,. \tag{5}$$

Since most authors use (4) when working with finite sums [2,3], I will demonstrate its use first. Let $P(n)$ be the proposition that

$$1^2 + 2^2 + 3^3 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1) \tag{6}$$

for all $n \in Z^+$ (note: $m = 1$). Let $L(n) = 1^2 + 2^2 + 3^3 + \cdots + n^2$ and $R(n) = \frac{1}{6}n(n+1)(2n+1)$. Since $L(1) = 1 = R(1)$ then $P(1)$ is true. Now,

$$R(k+1) = \tfrac{1}{6}(k+1)(k+2)(2k+3)$$

and

$$\begin{aligned} R(k) + a_{k+1} &= \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \tfrac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \tfrac{1}{6}(k+1)[2k^2 + 7k + 6] \\ &= \tfrac{1}{6}(k+1)(2k+3)(k+2) = R(k+1)\,. \end{aligned}$$

Thus, by Theorem 1 we know the $P(n)$ is true.

To prove the same result using (5) instead, we compare quantities a_{k+1} and $R(k+1) - R(k)$. Now

$$a_{k+1} = (k+1)^2$$

and

$$\begin{aligned} R(k+1) - R(k) &= \tfrac{1}{6}(k+1)(k+2)(2k+3) - \tfrac{1}{6}k(k+1)(2k+1) \\ &= \tfrac{1}{6}(k+1)[2k^2 + 7k + 6 - 2k^2 - k] \\ &= (k+1)^2 \end{aligned}$$

Therefore $P(k+1)$ and (consequently) $P(n)$ are both true.

It is clear that for finite-sum problems, $P(k+1)$ is generally more easily verified (or else rejected) by use of (5) rather than (4) for the simple reason that by using (5) all we need do is to simplify and compare the two differences $L(k+1) - L(k) = a_{k+1}$ and $R(k+1) - R(k)$; whereas, using (4), we must "work a_{k+1} into $R(k)$ in such a way as to produce *if possible* $R(k+1)$." Polya [4] is one of the few authors who recommends our "difference" approach, though he has not developed it as far as we have.

Theorem 1 is not convenient in its present form to use on the class of problems where $L(n)$ is a finite product. To develop algebraic sentences equivalent to $P(k)$ and $P(k+1)$ for this class of problems we proceed as follows. Let $L(n)$ be the finite product

$$L(n) = f_m f_{m+1} \cdots f_n \,, \tag{7}$$

where $n \geq m$ and $L(n)$ is never zero. We note that

$$f_{k+1} = \frac{L(k+1)}{L(k)} \,. \tag{8}$$

Omitting the details, we assume that $L(k) = R(k)$, then $L(k+1) = R(k+1)$ if and only if

$$R(k+1) = R(k)\frac{L(k+1)}{L(k)} = R(k) f_{k+1} \,. \tag{9}$$

THEOREM 2 Let $P(n)$ be the proposition that $L(n) = R(n)$ for all integers such that $n \geq m$, where $L(n)$ is given by (7), and if $P(m)$ is true by demonstration, and $P(k)$ is assumed true for any $k \geq m$, then $P(k+1)$ and (consequently) $P(n)$ are both true if and only if

$$R(k+1) = R(k) f_{k+1} \,, \tag{10}$$

or equivalently, if and only if

$$f_{k+1} = \frac{R(k+1)}{R(k)} \,. \tag{11}$$

As an example of this method, let $P(n)$ be the proposition that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \tag{12}$$

for all integers $n \geq 2$ (note: $m = 2$). Since $L(2) = 3/4 = R(2)$ then $P(2)$ is true. Now,

$$f_{k+1} = 1 - \frac{1}{(k+1)^2}$$

$$\begin{aligned}\frac{R(k+1)}{R(k)} &= \frac{(k+2)/2(k+1)}{(k+1)/2k} \\ &= \frac{k^2+2k}{(k+1)^2} \overset{\text{VE}}{=} \frac{(k^2+2k+1)-1}{(k+1)^2} = 1 - \frac{1}{(k+1)^2}\end{aligned}$$

Therefore $P(n)$ is true.

3 Conclusion

By introducting the abstract functions $L(n)$ and $R(n)$ we can specify simple algebraic sentences which, if true, are sufficient to establish the validity of $P(n)$ for whole classes of induction problems. We have developed the operational equations (5) and (11) for finite sums and finite products, respectively; their use should allow for a significant reduction in the algebraic complexity of the solution compared to their conventional counterparts (4) and (10).

[1] Saracino, D. 1980. *Abstract Algebra: A First Course.* Massachusetts: Addison-Wesley Publishing Company.
[2] Hillman, A. P. and Alexanderson, G. L. 1978. *A First Course in Abstract Algebra*, 2nd ed. Belmont, California: Wadsforth Publishing Company.
[3] Swokowski, E. 1984. *Calculus with Analytic Geometry.* 3rd ed. pp. A1–A5. Massachusetts: Prindle, Weber, and Schmidt Publishers.
[4] Polya, G. 1965. *Induction and Analogy in Mathematics*, 5th ed. pp. 108–110. Princeton University Press.