# The hyperintegers of ordinality $\omega$

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**Abstract**

This paper is a reprint of an article that first appeared in the *Arizona Journal of Natural Philosophy*, February, 1987. Hyperintegers, being extensions of the integers, are infinite strings of digits with suitable definitions of addition and multiplication to form a ring. This ring has four idempotents, two of which are infinite, nonrepeating sequences of digits.

## 1 Introduction

The numeral form of integers most commonly used is an ordered string of digits which have a rightmost digit, a next-to-rightmost digit, and so on, for a finite number of digits. Furthermore, it has a sign (or lack of one) that which indicates whether the integer is positive or negative. Each place in the numeral represents a power of 10, so that each numeral form of an integer also represents a cardinal number. Addition and multiplication operations, which have long been defined on these numerals, are intentionally isomorphic to the actual addition and multiplication of cardinal number themselves. We shall refer to these two operations as *long-hand addition* and *long-hand multiplication*. For each of these numerals there corresponds a unique, ordered, finite set of digits (in the form of digital strings), where the first element of the set is the rightmost digit of the numeral, the second is the next-to-rightmost digit, and so on.

Nevertheless, these digital numerals are numbers in their own right, quite apart from their traditional association as symbols for cardinal numbers; they are, in fact, finite ordinal numbers. We shall drop this limiting association with cardinal numbers, and thus free ourselves to consider extending our definition of addition and multiplication to infinite, ordered, digital strings. And having done this, we find the intriguing result that two of these ordinal numbers are infinite idempotents. Note: the hyperintegers introduced here are not related to nonstandard analysis, but to the 10-adic integers.

## 2 Transfinite digital number system

We define a *transfinite digital* as an ordered set of digits of cardinality $\aleph_0$. Of the infinite number of possible transfinite digitals, this paper will focus on those of ordinality $\omega$, as defined by Cantor [1]. Thus, every transfinite digital has the properties: 1) of having a first (or rightmost) digit, 2) that each of its digits has an immediate successor, 3) that each of its digits except the first has an immediate predecessor, and 4) that it has no last digit. By *digit* we mean any one of the cardinal numbers from 0 to 9, inclusive, the set of which we represent by $\mathcal{D}$. The numeral form of a transfinite digital is formed by taking as its rightmost digit the first digit of the number, and for its next-to-right most digit we take the second digit, and so on. Since we cannot write down every digit of a transfinite digital, then we will employ short-hand notations for infinite continuations, such as the ellipsis "…"

We will allow the transfinite digitals to be formally positive or negative without implying that signed transfinite digitals are ordered for all transfinite digitals. The set of all positive transfinite digitals we refer to as $\mathcal{T}^+$, and the set of all negative transfinite digitals as $\mathcal{T}^-$. The union of these two sets we refer to as $\mathcal{T}$. Therefore

$$\mathcal{T} = \mathcal{T}^+ + \mathcal{T}^- . \quad (1)$$

For all transfinite digitals $x$, we define the function $p_i(x)$ to be the $i$th digit as indexed from the rightmost digit, which is $p_1(x)$. Thus, $p_i(x)$ is the digit in the $i$th component of $x$.

Let $x, y$ be any two elements of $\mathcal{T}$; then $x$ is said to be equal to $y$ if and only if

1) $x$ and $y$ have the same sign, and
2) $p_i(x) = p_i(y)$ for all $i \in \mathbb{Z}^+$

where $\mathbb{Z}^+$ are the positive integers. This definition has one exception: we will make no distinction between the positive and negative transfinite digital whose every digit is 0.

## 3 Definitions of binary operations of $\mathcal{T}^+$

We define the following addition operation on two arbitrary positive transfinite digitals (TD's) $x$ and $y$, denoting their sum as $z$.

1) initialize the digit-valued variable *carry* to 0,
2) initialize $i$ to 1,
3) set $p_i(z) = [p_i(x) + p_i(y) + carry] \mod 10$,
4) $carry = [p_i(x) + p_i(y) + carry - p_i(z)]/10$,
5) increment $i$ by 1,
6) repeat steps 3) to 5) $N - 1$ times to obtain $N$ digits of the sum.

Thus, digit-wise addition, being an extension of long-hand addition, is a rule for adding infinite strings of digits. And these TD's formed as a sum are well defined, since any particular digit can be calculated in principle. Similarly, the algorithm for digit-wise multiplication is the same as for long-hand multiplication with the exception that it is an operation on infinite strings of digits. Furthermore, *digit-wise subtraction* is the extension of the subtraction of two positive finite digit strings, to the subtraction of two infinite digital strings.

Clearly, the addition or multiplication of two positive TD's is also a TD, and we will show that the difference is too. Let $x$ and $y$ be elements of $\mathcal{T}^+$; then $x - y$ will also be in $\mathcal{T}^+$ because digit-wise subtraction permits the borrowing of 10 from its successor, or even from an infinite number of left-wise successors. We demonstrate: What is the difference $1 - 3$ in the set $\mathcal{T}^+$? By borrowing a 10 from infinity, sotospeak, we get

$$\ldots 999^1 1 - \ldots 0003 = \ldots 9998\,.$$

We can test our solution for accuracy by adding $\ldots 003$ to $\ldots 9998$ to get $\ldots 0001$.

We say that $x$ is a *terminating* TD if there exists an $n \in \mathbb{Z}^+$ such that $p_i(x) = 0$ for all $i \geq n$. The set of all terminating TD's we call the *digital integers*, which we denote by $\mathcal{Z}$. The set of all positive and negative nonterminating TD's we call the *pseudointegers* and refer to them by $\mathcal{P}$. Clearly, $\mathcal{Z}$ forms a ring isomorphic to the ring of integers $\{\mathbb{Z}, +, \cdot\}$. We will refer to the set of $\mathbb{Z}$ as the *inductive* integers. Thus, we have that $\mathcal{P} \cup \mathcal{Z} = \mathcal{T}$.

**Numeral conventions**

A bar placed over a digit or string of digits means that the digit or string repeats indefinitely, as in the following examples:

$$\overline{9} = \ldots 999$$
$$\overline{5412} = \ldots\ 5412\,5412$$

The two numbers $\overline{9}$ and $\overline{5412}$ are examples of a special type of TD and are said to be *repeating* or *rational* in $\mathcal{T}$. In general we write $\overline{z} = \ldots zzz$, where $z$ is a finite digital string of $s$ digits. We say that $s$ is the *span* of $z$. A transfinite digital that cannot be represented as an infinite sequence of repeating digital strings is said to be *nonrepeating* or *irrational* in $\mathcal{T}$. A repeating TD of the form $\overline{z}$ is said to be *primitive*, otherwise the TD has the form $\overline{z} + a$, where $a$ is a nonzero, finite digital string, and is said to be *nonprimitive*.

In $\mathcal{T}$ multiplying a TD by $10^n$ is an operation that places $n$ 0's at the beginning of the TD. For example, $10^3 \cdot \overline{9} = \overline{9}000$.

Let $n \in \mathbb{Z}^+$ and $x \in \mathcal{T}^+$, then $x^n$ is $n$ factors of $x$, and means just the usual exponentiation. We define the zero of the TD's to be the TD whose every digit is 0, and clearly this element is the additive identity. The number $\overline{0}1$ is clearly the multiplicative identity for the TD's. The zero and unity in $\mathcal{T}^+$, though properly written as $\overline{0}$ and $\overline{0}1$ respectively, will be written simply as 0 and 1 with context to make their TD status evident.

## 4 Definition of complement

Let the Roman numeral X be a single-letter representation of the cardinal number 10. For computational convenience we define the hybrid numeral $\overline{9}X$ by

$$0 = \ldots 999 + \overline{0}1 = \ldots 999\mathrm{X} = \overline{9}\mathrm{X}\,. \tag{2}$$

Thus, the symbols 0, $\overline{0}$, and $\overline{9}$X all representing the same TD $\overline{0}$. The *complement* of a TD $x$ is given by

$$\mathrm{comp}(x) = \begin{cases} \overline{9}\mathrm{X} - x & \text{for } x \text{ positive,} \\ -(\overline{9}\mathrm{X} + x) & \text{for } x \text{ negative.} \end{cases} \tag{3}$$

Or, put another way,

$$\text{for } x \text{ positive} \quad \begin{cases} p_1(\mathrm{comp}(x)) = \mathrm{X} - p_1(x) \\ p_i(\mathrm{comp}(x)) = 9 - p_i(x) \quad \text{for } i \geq 2\,. \end{cases}$$

When $x$ is negative we may evaluate its complement by finding the comp$(-x)$ and then take the negative of that result. Thus, every positive TD can be written as a negative TD, and vice versa.

From the definition of the complement function we obtain the following result

$$x = -\mathrm{comp}(x) = -(\overline{9}\mathrm{X} - x)\,. \tag{4}$$

For example,

$$\begin{aligned} 13 &= -\mathrm{comp}(13) = -(\overline{9}\mathrm{X} - 13) = -\ldots 99987 \\ -101 &= -\mathrm{comp}(101) = -(\overline{9}\mathrm{X} + 101) = \overline{9}\mathrm{X} - 101 = \overline{9}899\,. \end{aligned}$$

Let $x$ be any positive TD, then

$$x + \mathrm{comp}(x) = 0\,. \tag{5}$$

The proof of this follows immediately from the definition of the complement function and digit-wise addition and equality. Thus, for all $x \in \mathcal{T}^+$, comp$(x)$ acts as an additive inverse of $x$. It is easy to show that comp(comp$(x)) = x$.

From the foregoing we conclude that $(\mathcal{T}^+, +, \cdot)$ is a ring with unity.

## 5 The hyperintegers

The complement function allows us to equate for all $x \in \mathcal{T}^+$ a negative TD by $x = -\mathrm{comp}(x)$. We take as the hyperintegers the set of all such cosets $\{x, -\mathrm{comp}(x)\}$ where $x \in \mathcal{T}^+$, and we represent this set as $\mathcal{H}$.

Now we introduce a useful function on $\mathcal{H}$. Let $x \in \mathcal{H}$, then

$$r_n(x) \equiv \pm \ldots 0\,0\,0\,p_n(x)\,p_{n-1}(x)\,\ldots\,p_1(x)\,.$$

We can interpret the $r_n$ function as selecting the $n$ rightmost digits of a TD, retaining sign as well. This function allows for an alternative definition of equality in $\mathcal{T}^+$. Let $x, y$ be any two elements of $\mathcal{T}^+$, then $x$ and $y$ are said to be *equal* if and only if

$$r_i(x) = r_i(y) \qquad \text{for all } i \in \mathbb{Z}^+ .$$

From here on, the set of TD's is assumed to be constrained by Equation (5), i.e., TD's are just hyperintegers.

Let $\mathrm{T}^n$ be the set of all TD's $\mod 10^n$ for any $n \in \mathbb{Z}^+$. Then $(\mathrm{T}^n, +, \cdot)$ is a ring with addition and multiplication taken $\mod 10^n$. We refer to $\mathrm{T}^n$ as the *truncation ring of order* $n$. The function $r_n()$ is an onto function from hyperintegers to $\mathrm{T}^n$, with the properties:

$$r_n(r_n(x)) = r_n(x) \tag{6a}$$
$$r_n(x + y) = r_n(r_n(x) + r_n(y)) = r_n(x) + r_n(y) \tag{6b}$$
$$r_n(x \cdot y) = r_n(r_n(x) \cdot r_n(y)) = r_n(x) \cdot r_n(y) . \tag{6c}$$

Again, it is understood that the addition and multiplication on the right sides of (6b) and (6c) are taken $\mod 10^n$. Obviously, every TD that's a multiple of $10^n$ is in the kernel of $r_n$. For future convenience we adopt the notation $x_{,n} = r_n(x)$. From (6a) we see that $r_n$ is an idempotent operator.

Before we get to our main results we need to define an idempotent in $\mathrm{T}^+$: idempotents are numbers that square to be themselves. In our current number system that requires either

$$p_n(x^2) = p_n(x) \qquad \text{for all } n \in \mathbb{Z}^+ \tag{7}$$

or

$$r_n(x^2) = r_n(x) \qquad \text{for all } n \in \mathbb{Z}^+ .. \tag{8}$$

# 6 Main results

LEMMA 1:

There is no TD solution to (8) whose first digit is in the set $\{2, 3, 4, 7, 8, 9\}$. Since the proofs for each case is similar to every other case, we shall prove this for one case only. Let $x$ be any element of $\mathcal{T}^+$ such that $r_1(x) = 7$. Now $x$ can be written in the form $10x' + 7$. But $r_1((10x' + 7)^2) = r_1(\ldots 49) = 9 \neq 7$. Thus, there is no idempotent that starts with 7.

LEMMA 2:

The only TD in $\mathrm{T}^+$ whose first element is 0 that satisfies (8) is the TD $\overline{0}$, or zero. Assume not. Let $y$ be a positive TD not equal to zero but whose first digit is 0. Then there is some $i \geq 2$ such that $p_i(y) \neq 0$. Without loss of generality we assume that this $i$ is the smallest value whose place value is different from 0. Therefore $r_{i-1}(y)$ is $i-1$ 0's, and $r_{2(i-1)}(y^2)$ is $2(i-1)$ 0's, implying that $p_i(y^2) = 0$. But this contradicts our assumption that $p_n(y^2) = p_n(y)$ for all $n \in \mathbb{Z}^+$, since $p_i(y^2) \neq p_i(y)$.

LEMMA 3:

Unity is the only positive TD that starts with the digit 1 that is idempotent. Assume not. Obviously, unity satisfies the idempotency condition, thus we need only show uniqueness. Let $y$ be an assumed idempotent of $\mathcal{T}^+$ which is not unity but does start with the digit 1. Let $n$ be the smallest integer such that $p_n(y) \neq 0$. Let $d = p_n(y)$. Then,

$$r_n(y) = d10^{n-1} + 1\,.$$

And for $y$ to satisfy (8) it must be true that

$$\begin{aligned} r_n([d10^{n-1}+1]^2) &= r_n(d^2 10^{2(n-1)}) + r_n(2d10^{n-1}) + 1 \\ &= 2d10^{n-1} + 1\,, \end{aligned}$$

since $d^2 10^{2(n-1)}$ is in $\ker(r_n)$. Therefore $2d10^{n-1} = d10^{n-1}$, implying that $d = 0$, contradicting our assumption.

LEMMA 4:

There exits a unique positive TD whose first digit is 5 which satisfies (8).

Part 1: Existence.[1]

Our approach will be to specify a particular positive TD by a recursive definition, and then to prove that it satisfies (8). Let $\mathrm{e}_1$ be the TD defined by the following definition:

$$r_1(\mathrm{e}_1) = 5 \tag{9a}$$

$$r_{n+1}(\mathrm{e}_1) = r_{n+1}([\mathrm{e}_{1,n}]^2) \qquad \text{for all } n \in \mathbb{Z}^+\,. \tag{9b}$$

We'll now prove that $\mathrm{e}_1$ satisfies (8) by an induction proof. It's easy to see that (8) is satisfied by $\mathrm{e}_1$ for $n = 1$. Next we assume that (8) holds for some arbitrary $k$, and then show that it must also hold for case $k+1$. Thus, we want to show that

$$r_{k+1}(\mathrm{e}_1^2) = r_{k+1}(\mathrm{e}_1)\,. \tag{10}$$

Now

$$r_{k+1}(\mathrm{e}_1^2) = r_{k+1}([\mathrm{e}_{1,k+1}]^2) \tag{11}$$

by (6c). Let $d = p_{k+1}(\mathrm{e}_{1,k+1})$, then

$$\mathrm{e}_{1,k+1} = d10^k + \mathrm{e}_{1,k} \tag{11}$$

[1] For an alternative proof see Appendix 1.

And

$$
\begin{aligned}
r_{k+1}(\mathrm{e}_1^2) &= r_{k+1}([\mathrm{e}_{1,k+1}]^2) \\
&= r_{k+1}(d^2 10^{2k} + 2d10^k \mathrm{e}_{1,k} + [\mathrm{e}_{1,k}]^2) \\
&= r_{k+1}([\mathrm{e}_{1,k}]^2) \\
&= r_{k+1}(\mathrm{e}_1)\,. 
\end{aligned} \tag{12}
$$

Where we used in the third step the fact that $d^2 10^{2k}$ and $2d10^k \mathrm{e}_{1,k}$ are in the kernel of $r_{k+1}$. (Note that $10\,|\,2\mathrm{e}_{1,k}$.) And in the last step we used the definition of $\mathrm{e}_1$ from (9b).

Part 2: Uniqueness. This is left to the reader as an easy exercise that follows the same methods as do our previous uniqueness proofs.

The first 100 digits of $\mathrm{e}_1$ are:

100–51  3953007319 1081698029 3850989006 2166509580 8638110005

50–01  5742342323 0896109004 1066199773 9225625991 8212890625

LEMMA 5:

One can easily show that a second nontrivial idempotent can be found from $\mathrm{e}_1$, namely $\mathrm{e}_2 \equiv 1 - \mathrm{e}_1$. Then we have that

$$\mathrm{e}_1 + \mathrm{e}_2 = 1\,. \tag{13}$$

But what does $\mathrm{e}_2$ look like?

$$
\begin{aligned}
\mathrm{e}_2 &= 1 - \mathrm{e}_1 = 1 + \mathrm{comp}(\mathrm{e}_1) \\
&= 1 + \ldots 375 = \ldots 376
\end{aligned}
$$

Therefore the first digit of $\mathrm{e}_2$ starts with a 6. And since the idempotents starting with 6 are paired to the idempotents starting with 5, and since there is only one idempotent starting with 5, then there is only one idempotent starting with 6.

The first 100 digits of $\mathrm{e}_2$ are:

100–51  6046992680 8918301970 6149010993 7833490419 1361889994

50–01  4257657676 9103890995 8933800226 0774374008 1787109376

It is easy to prove that $\mathrm{e}_1\mathrm{e}_2 = 0$, and the reader can test this for a few rightmost digits of each idempotent.

# 7 Conclusion

Thus, there are four distinct idempotents in the hyperintegers: $\{0, -0\}$, $\{1, -\overline{9}\}$, $\{\mathrm{e}_1, -\mathrm{comp}(\mathrm{e}_1)\}$, $\{\mathrm{e}_2, -\mathrm{comp}(\mathrm{e}_2)\}$. Also, unity has four square roots, namely,

$\pm 1$ and $\pm\sigma$, where $\sigma = \mathrm{e}_1 - \mathrm{e}_2$. We can use the nontrivial idempotents to generate the principal ideals

$$I_k \equiv \mathcal{H}\mathrm{e}_k = \{h\mathrm{e}_k : h \in \mathcal{H}\} \qquad k = 1, 2\,. \tag{14}$$

Therefore $\mathcal{H}$ can be expressed as the direct sum of these two ideals:

$$\mathcal{H} = I_1 \oplus I_2\,. \tag{15}$$

# 8 Appendix 1

This alternative proof attempts to make the inductive hypothesis

$$r_k(\mathrm{e}_1^2) = r_k(\mathrm{e}_1)\,. \tag{A.1a}$$

clearly used in the proof. Equation (A.1a) implies that

$$r_k([\mathrm{e}_{1,k}]^2) = r_k(\mathrm{e}_{1,k})\,. \tag{A.1b}$$

Much of the former proof is the same here. This time, however, we take a different recursive definition to generate the presumed idempotent. Let $\mathrm{e}_1$ be the TD defined by the following:

$$p_1(\mathrm{e}_1) = 5\,, \tag{A.2a}$$
$$p_{n+1}(\mathrm{e}_1) = p_{n+1}([\mathrm{e}_{1,n}]^2) \qquad \text{for all } n \in \mathbb{Z}^+\,. \tag{A.2b}$$

We'll now prove that $\mathrm{e}_1$ satisfies (8) by an induction proof, taking (A.1a) as our inductive hypothesis. It's easy to see that (8) is satisfied by $\mathrm{e}_1$ for $n = 1$. Next we assume that (8) holds for some arbitrary $k$ as in (A.1a,A.1b), and then show that it must also hold for case $k + 1$. Thus, we want to show that

$$r_{k+1}(\mathrm{e}_1^2) = r_{k+1}(\mathrm{e}_1)\,. \tag{A.3}$$

Now

$$r_{k+1}(\mathrm{e}_1^2) = r_{k+1}([\mathrm{e}_{1,k+1}]^2)$$

by (6c). Let $d = p_{k+1}(\mathrm{e}_{1,k+1})$, then

$$\mathrm{e}_{1,k+1} = d10^k + \mathrm{e}_{1,k}\,. \tag{A.4}$$

And

$$\begin{aligned}
r_{k+1}(\mathrm{e}_1^2) &= r_{k+1}(d^2 10^{2k} + 2d10^k \mathrm{e}_{1,k} + [\mathrm{e}_{1,k}]^2) \\
&= p_{k+1}([\mathrm{e}_{1,k}]^2)10^k + r_k([\mathrm{e}_{1,k}]^2) \\
&= p_{k+1}([\mathrm{e}_{1,k}]^2)10^k + r_k(\mathrm{e}_{1,k}) \quad \text{by the inductive hypothesis} \\
&= p_{k+1}(\mathrm{e}_1)10^k + r_k(\mathrm{e}_1) \quad \text{by defn. of } \mathrm{e}_1 \text{ in (A.2b) and by (6a)} \\
&= r_{k+1}(\mathrm{e}_1)\,.
\end{aligned} \tag{A.5}$$

[1] G. Cantor. *Contributions to the Founding of the Theory of Transfinite numbers*. Dover. New York (p. 56–57).