A special vector calculus identity

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Abstract

Here we'll use Gibbs's vector calculus to prove some additional results in vector calculus.

1 Introduction

Let $f(x, y, z)$ and $g(x, y, z)$ be scalar fields with continuous second partials everywhere.

Compute $\nabla \cdot (\nabla f \times \nabla g)$, where the symbol $\times$ means the cross product.

Note: For vectors A and B, $A \times B = e_j \epsilon_{jkm} A_k B_m$, with summation on repeated indices and e_j is the jth member of the set of three orthonormal unit vectors. In other words, each vector A can be written as

$$A = A_i e_i, \quad i = 1, 2, 3. \tag{1}$$

Note: $\epsilon_{jkm} = +1$ if jkm is an even permutation of 123, and is $= -1$ if it is an odd permutation of $1, 2, 3$ and $=$ zero otherwise. So, ϵ_{jkm} is antisymmetric for all pairs of its subscripts.

For the problem to show that $\nabla \cdot [(\nabla f) \times (\nabla g)] = 0$, we need some other results. One of which is this

$$e_i \cdot e_j = \delta_{ij} \,, \tag{2}$$

where δ_{ij} is the Kronecker delta, and $\delta_{ij} = 1$ if $i = j$, and equal zero otherwise. Also,

$$\nabla \cdot Q = (e_i \partial_i) \cdot (Q_j e_j) \,, \tag{3}$$

where we used different subscripts because they are summed up independently of each other. Okay, once more,

$$\begin{aligned} \nabla \cdot Q &= (e_i \partial_i) \cdot (Q_j e_j) \\ &= (e_i \cdot e_j) \partial_i Q_j \\ &= \delta_{ij} \partial_i Q_j \\ &= \partial_i Q_i \,. \end{aligned} \tag{4}$$

The summed on subscripts are said to be "dummy" because they can be replaced by any other index so long as that index is not already in use in the same term. Just one more prior result to go:

Let $[M_{ij}]$ be any symmetric 3×3 matrix. Then

$$\epsilon_{ijk} M_{ij} = 0 \,, \tag{5}$$

with summing on indices i, j. Prove this to your satisfaction.

Now we're ready.

$$
\begin{aligned}
\nabla \cdot [(\nabla f) \times (\nabla g)] &= e_i \partial_i \cdot [e_j \epsilon_{jkm} (\partial_k f)(\partial_m g)] \\
&= e_i . e_j \partial_i [\epsilon_{jkm} (\partial_k f)(\partial_m g)] \\
&= \delta_{ij} \partial_i [\epsilon_{jkm} (\partial_k f)(\partial_m g)] \\
&= \epsilon_{jkm} [(\partial_j \partial_k f)(\partial_m g) + (\partial_k f)(\partial_j \partial_m g)] \\
&= (\epsilon_{jkm} \partial_j \partial_k f)(\partial_m g) + (\partial_k f)(\epsilon_{jkm} \partial_j \partial_m g) \\
&= 0\,. \qquad (6)
\end{aligned}
$$

We got zero because both terms on the right above are zero. I'll prove this for the first term only though. Because f and g have continuous second partials everywhere, the order of their partial derivatives is irrelevant. So, for the case of the derivative of f

$$\partial_j \partial_k f = \partial_k \partial_j f\,. \qquad (7)$$

This means that the 3×3 matrix of derivatives of f, $[\partial_j \partial_k f]$, is symmetric. So, by the lemma above

$$\epsilon_{jkm}(\partial_j \partial_k f) = 0 \quad (m = 1, 2, 3)\,. \qquad (8)$$

Done. This proof employs just about every thing you'll need to know to use cartesian tensors in advanced calculus.

The only other result that commonly comes up in cartesian tensors is the following

$$\epsilon_{ijk} \epsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}\,. \qquad (9)$$

You can investigate this further from Wikipedia or elsewhere if you want.