The Integration the Jacobi Elliptic Functions

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Abstract

Herein are presented techniques for the integration of the Jacobi elliptic functions (JEF).

1 Introduction

The integration of function of the Jacobi elliptic functions (JEF) is usually more difficult than the integration of their trigonometric counterparts because the derivatives of the JEF are more complicated than those of trigonometric functions. There are already some well-known transformations on the JEF, including the *Jacobi imaginary transformation*, which replaces the integrals in JEFs with integrals in ordinary trigonometric functions. (This paper was first published by the same author in the *The Arizona Journal of Natural Philosophy*, Sept. 1988, and reprinted in July 1996).

2 Main

The JEF have two basic equations which inter-relate them:

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1 \tag{1}$$

$$k^2 \mathrm{sn}^2 u + \mathrm{dn}^2 u = 1 \tag{2}$$

where k, the *modulus*, is constrained by $0 < k < 1$. The derivatives of these functions are:

$$\frac{d}{du}\mathrm{sn}\, u = \mathrm{cn}\, u\, \mathrm{dn}\, u\,, \tag{3}$$

$$\frac{d}{du}\mathrm{cn}\, u = -\mathrm{sn}\, u\, \mathrm{dn}\, u\,, \tag{4}$$

$$\frac{d}{du}\mathrm{dn}\, u = -k^2\mathrm{sn}\, u\, \mathrm{cn}\, u\,. \tag{5}$$

We now define a transformation on the JEF defined by the triangle in Fig. 1, from which we arrive at the following transformation equations:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\mathrm{sn}\, u &=& \sin\phi & \sin\phi &=& \mathrm{sn}\, u \\
\mathrm{cn}\, u &=& \cos\phi & \cos\phi &=& \mathrm{cn}\, u \\
\mathrm{dn}\, u &=& \Delta\phi & \Delta\phi &=& \mathrm{dn}\, u \\
du &=& d\phi/\Delta\phi & d\phi &=& \mathrm{dn}\, u\, du
\end{array} \tag{6}$$

where $\Delta\phi \equiv (1 - k^2 \sin^2\phi)^{\frac{1}{2}}$.

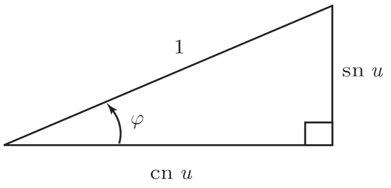

Fig. 1

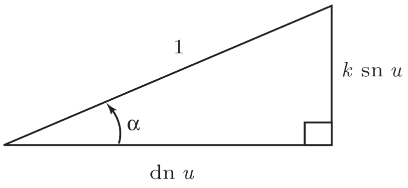

Fig. 2

Although we have treated the angle u as fundamental, and introduced ϕ as a transformation on u, historically, the reverse is true. Most texts [1] that mention the JEF will also give a description of ϕ to u. In short, they are related by the following:

$$\mathrm{sn}\, u \ \sin \mathrm{am}\, u = \sin\phi\,, \tag{7a}$$

$$\mathrm{cn}\, u \ \cos \mathrm{am}\ u = \cos\phi\,, \tag{7b}$$

where $\phi = \mathrm{am}\ u$ (the "amplitude" of u, which is a function of u whose specific form does not concern us at this point).

Using (6), the following integration is easily performed.

$$\begin{aligned}\int \mathrm{sn}^{\,2} u\, \mathrm{dn}\, u\, du = \int \sin^2\phi\, d\phi &= \tfrac{1}{2}(\phi - \sin\phi\cos\phi) \\ &= \tfrac{1}{2}(\mathrm{am}\ u - \mathrm{sn}\, u\, \mathrm{cn}\, u)\,.\end{aligned} \tag{8}$$

From (2) we form an α transformation as represented in Fig. 2, and given by the following transformation equations:

$$\begin{array}{rcl@{\qquad}rcl} \mathrm{sn}\, u &=& k^{-1}\sin\alpha & \sin\alpha &=& k\,\mathrm{sn}\, u \\ \mathrm{cn}\, u &=& \nabla\alpha & \cos\alpha &=& \mathrm{dn}\, u \\ \mathrm{dn}\, u &=& \cos\alpha & \Delta\alpha &=& \mathrm{cn}\, u \\ du &=& d\alpha/k\,\nabla\alpha & d\alpha &=& k\,\mathrm{cn}\, u\, du \end{array} \tag{9}$$

where $\nabla\alpha \equiv (1 - k^{-2}\sin^2\alpha)^{\frac{1}{2}}$. Therefore, to integrate $\int \mathrm{sn}^{\,2} u\, \mathrm{cn}\, u\ du$, we have

$$\begin{aligned}\int \mathrm{sn}^{\,2} u\, \mathrm{cn}\, u\, du = k^{-3}\int \sin^2\alpha\, d\alpha &= \tfrac{1}{2}k^{-3}(\alpha - \sin\alpha\cos\alpha) \\ &= \tfrac{1}{2}k^{-3}[\,\sin^{-1}(k\,\mathrm{sn}\, u) - k\,\mathrm{sn}\, u\, \mathrm{dn}\, u)\,]\end{aligned} \tag{10}$$

By defining the *complementary modulus* $k' \equiv (1-k^2)^{\frac{1}{2}}$ and $\epsilon = k'/k$, then combining (1) and (2) we have

$$\mathrm{dn}^{\,2} u - k^2\mathrm{cn}^{\,2} u = k'^2 \tag{11}$$

or

$$(1/k'^2)\,\mathrm{dn}^{\,2} u - (1/\epsilon^2)\,\mathrm{cn}^{\,2} u = 1\,. \tag{12}$$

Now we could transform this last equation by the hyperbolic functions

$$\cosh^2\beta = (1/k'^2)\,\mathrm{dn}^{\,2} u \qquad \text{and} \qquad \sinh^2\beta = (1/\epsilon^2)\,\mathrm{cn}^{\,2} u\,, \tag{13}$$

but instead we will use the following "imaginary" transformation:

$$\begin{array}{rcl@{\qquad}rcl} \mathrm{sn}\, u &=& \nabla_+\beta & \sin\beta &=& (k\,\mathrm{cn}\, u)/i\epsilon \\ \mathrm{cn}\, u &=& i\epsilon\,\sin\beta & \cos\beta &=& (\mathrm{dn}\, u)/k' \\ \mathrm{dn}\, u &=& k'\cos\beta & \nabla_+\beta &=& \mathrm{sn}\, u \\ du &=& d\beta/ik\,\nabla_+\beta & d\beta &=& ik\,\mathrm{sn}\, u\, du \end{array} \tag{14}$$

where $\nabla_+\beta \equiv (1+\epsilon^2\sin^2\beta)^{\frac{1}{2}} = k^{-1}(1-k'^2\cos^2\beta)^{\frac{1}{2}}$.

Consider the simplicity of the following integration

$$\begin{aligned}\int\frac{du}{\operatorname{sn}u} &= \frac{1}{ik}\int\frac{d\beta}{\nabla_+\beta} = \frac{k}{i}\int\frac{d\beta}{1-k'^2\cos^2\beta}\\ &= \frac{1}{i}\tan^{-1}\left[\frac{\tan\beta}{k}\right] = \frac{1}{i}\tan^{-1}\left[\frac{\operatorname{cn}u}{i\operatorname{dn}u}\right]\,. \end{aligned}\tag{15}$$

This last integration is easier still by using the η transformation introduced later on.

Now, dividing (1) by $\operatorname{cn}u$ and rewriting, we get

$$\left(\frac{1}{\operatorname{cn}u}\right)^2 - \left(\frac{\operatorname{sn}u}{\operatorname{cn}u}\right)^2 = 1\,.\tag{16}$$

We derive the so-called *Jacobi imaginary transformation* by the following equations:

$$\begin{array}{rcl@{\qquad\qquad}rcl}\operatorname{sn}u &=& (1/i)\tan\theta & \sin\theta &=& i\operatorname{tn}u\\ \operatorname{cn}u &=& \sec\theta & \cos\theta &=& 1/\operatorname{cn}u\\ \operatorname{dn}u &=& \sec\theta\Delta'\theta & \Delta'\theta &=& \operatorname{dn}u/\operatorname{cn}u\\ du &=& d\theta/i\Delta'\theta & d\theta &=& i\operatorname{dn}u\,du/\operatorname{cn}u\end{array}\tag{17}$$

where $\Delta' \equiv (1-k'^2\sin^2\theta)^{\frac{1}{2}}$. As an example we have:

$$\begin{aligned}\int\frac{\operatorname{dn}u\,du}{\operatorname{sn}^2u\operatorname{cn}u} &= i\int\cot\theta\,d\theta = -i(\theta+\cot\theta)\\ &= \log\left(\frac{\operatorname{cn}u}{1-\operatorname{sn}u}\right) - \frac{1}{\operatorname{sn}u}\,.\end{aligned}\tag{18}$$

Subtracting (1) from (2) yields:

$$k'^2\operatorname{sn}^2u + \operatorname{cn}^2u = \operatorname{dn}^2u\,,\tag{19}$$

from which there follow the three equations

$$\left(\frac{k'\operatorname{sn}u}{\operatorname{dn}u}\right)^2 + \left(\frac{\operatorname{cn}u}{\operatorname{dn}u}\right)^2 = 1\,,\tag{20}$$

$$\left(\frac{\operatorname{dn}u}{\operatorname{cn}u}\right)^2 - \left(\frac{k'\operatorname{sn}u}{\operatorname{cn}u}\right)^2 = 1\,,\tag{21}$$

$$\left(\frac{\operatorname{dn}u}{k'\operatorname{sn}u}\right)^2 - \left(\frac{\operatorname{cn}u}{k'\operatorname{sn}u}\right)^2 = 1\,.\tag{22}$$

If in (20) we take $\cos\gamma = k'\operatorname{sn}u/\operatorname{dn}u$ and $\sin\gamma = i\operatorname{cn}u/\operatorname{dn}u$, then

$$\begin{array}{rcl@{\qquad\qquad}rcl}\operatorname{sn}u &=& \cos\gamma/\Delta\gamma & \sin\gamma &=& i\operatorname{cn}u/\operatorname{dn}u\\ \operatorname{cn}u &=& k'\sin\gamma/\Delta\gamma & \cos\gamma &=& k'\operatorname{sn}u/\operatorname{dn}u\\ \operatorname{dn}u &=& k'/\Delta\gamma & \Delta\gamma &=& k'/\operatorname{dn}u\\ du &=& -d\gamma/\Delta\gamma & d\gamma &=& -k'\,du/\operatorname{dn}u\end{array}\tag{23}$$

The transformations based on (1) and (2) together with the remaining ones based on (1), (2), and (11) and (12) are as follows:

$$\begin{array}{rcl@{\qquad\qquad}rcl}\operatorname{sn}u &=& 1/k\sin\tau & \sin\tau &=& 1/k\operatorname{sn}u\\ \operatorname{cn}u &=& i\Delta\tau/k\sin\tau & \cos\tau &=& \operatorname{dn}u/ik\operatorname{sn}u\\ \operatorname{dn}u &=& i\cot\tau & \Delta\tau &=& -i\operatorname{ctn}u\\ du &=& d\tau/\Delta\tau & d\tau &=& -i\operatorname{ctn}du\end{array}\tag{24}$$

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\operatorname{sn} u & = & \sec\zeta & \sin\zeta & = & \operatorname{cn} u/i \operatorname{sn} u \\
\operatorname{cn} u & = & \tan\zeta & \cos\zeta & = & 1/\operatorname{sn} u \\
\operatorname{dn} u & = & k' \sec\zeta \nabla'\zeta & \Delta\zeta & = & \operatorname{dn} u'/k' \operatorname{sn} u \\
du & = & d\zeta/ik' \nabla'\zeta & d\zeta & = & i \operatorname{dn} u \, du/\operatorname{sn} u
\end{array} \tag{25}$$

where $\nabla'\zeta = [\,1 - (1/k')^2 \sin^2\zeta\,]^{1/2}$.

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\operatorname{sn} u & = & i \sin\nu/k' \nabla'\nu & \sin\nu & = & k' \operatorname{sn} u/i \operatorname{cn} u \\
\operatorname{cn} u & = & 1/\nabla'\nu & \cos\nu & = & \operatorname{dn} u/\operatorname{cn} u \\
\operatorname{dn} u & = & \cos\nu/\nabla'\nu & \nabla'\nu & = & 1/\operatorname{cn} u \\
du & = & i \, d\nu/k' \nabla'\nu & d\nu & = & k' du/i \operatorname{cn} du
\end{array} \tag{26}$$

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\operatorname{sn} u & = & ik^{-1} \tan\xi & \sin\xi & = & k \operatorname{sn} u/i \operatorname{dn} u \\
\operatorname{cn} u & = & \nabla_+\xi/\cos\xi & \cos\xi & = & 1/\operatorname{dn} u \\
\operatorname{dn} u & = & \sec\xi & \nabla_+\xi & = & \operatorname{cn} u/\operatorname{dn} u \\
du & = & i \, d\xi/k \nabla_+\xi & d\xi & = & k \operatorname{cn} du/i \operatorname{dn} u
\end{array} \tag{27}$$

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\operatorname{sn} u & = & \sec\omega \nabla\omega & \sin\omega & = & k \operatorname{cn} u/\operatorname{dn} u \\
\operatorname{cn} u & = & \epsilon \tan\omega & \cos\omega & = & k'/\operatorname{dn} u \\
\operatorname{dn} u & = & k' \sec\omega & \Delta\omega & = & k' \operatorname{sn} u/\operatorname{dn} u \\
du & = & -d\omega/k \nabla\omega & d\omega & = & -kk' \operatorname{sn} u \, du/\operatorname{dn} u
\end{array} \tag{28}$$

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\operatorname{sn} u & = & \Delta\mu/k \cos\mu & \sin\mu & = & \operatorname{dn} u/k \operatorname{cn} u \\
\operatorname{cn} u & = & \epsilon/i \cos\mu & \cos\mu & = & \epsilon/i \operatorname{cn} u \\
\operatorname{dn} u & = & -ik' \tan\mu & \Delta\mu & = & k' \operatorname{sn} u/i \operatorname{cn} u \\
du & = & d\mu/\Delta\mu & d\mu & = & ik' \operatorname{sn} u \, du/\operatorname{cn} u
\end{array} \tag{29}$$

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\operatorname{sn} u & = & 1/\Delta'\eta & \sin\eta & = & i \operatorname{cn} u/k' \operatorname{sn} u \\
\operatorname{cn} u & = & k' \operatorname{sn}\eta/i\Delta'\eta & \cos\eta & = & \operatorname{dn} u/k' \operatorname{sn} u \\
\operatorname{dn} u & = & k' \cos\eta/\Delta'\eta & \Delta\eta & = & 1/\operatorname{sn} u \\
du & = & i \, d\eta/\Delta'\eta & d\eta & = & du/i \operatorname{sn} u
\end{array} \tag{30}$$

The set of transformations on the JEF presented above provide a large data base of transformations to simplify many integrals involving the JEF. The main advantage of this approach is that the transforms above not only transform the integrands of integrals but their differentials as well.

References

[1] H. Hancock, *Elliptic Integrals*, (Dover Publications, New York 1958), pp.24–30.