Integration Techniques Paper 1

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Abstract

Herein is a collection of integration techniques for indefinite integrals. I found the problems from a variety of sources, most of which I no longer know where they came from.

1 Introduction

Indefinite integrals, when they exist, result in functions, whereas, definite integrals, when they exist, result in numbers. The purpose of this paper is to present some techniques to perform indefinite integrals. My major strategy will be to reduce or convert a given integral into a form which is in a robust integral table, though, I may attempt to reduce the integral as far as I can take it. The integrals are presented in no particular order.

Warning: This paper assumes that the reader has a basic knowledge of integration (such as integration by parts), along with a familiarity with trigonometric and hyperbolic identities, logarithms, exponentials, partial fractions – generally speaking, the stuff found in a course on Algebra 2.

Note: The symbol D_x means to differentiate by x.

2 Reference Integrals and Derivatives for Later Use

First, the derivatives:

$$D_x \sin x = \cos x\,. \tag{1}$$

$$D_x \cos x = -\sin x\,. \tag{2}$$

$$D_x \sec x = \sec x \tan x\,. \tag{3}$$

$$D_x \tan x = \sec^2 x\,. \tag{4}$$

Now, the integrals:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\,. \tag{5}$$

$$\int \ln x\, dx = \ln x\,(x-1) + C\,. \tag{6}$$

$$\int \frac{du}{u} = \ln u + C\,. \tag{7}$$

$$\int \sec x\, dx = \ln\Big|\sec x + \tan x\Big| + C\,. \tag{8}$$

3 Virtual Emplacement

So much of mathematics employs tricks that get used over and over in a wide variety of subject areas, yet go unnamed, and thus are hard to explain to one's readers when one uses them. Decades ago I invented the term *virtual emplacement* to refer to the algebraic action of adding a zero to an expression, or multiplying or dividing an expression by unity, or more generally, performing some function and its inverse to an expression, such as

$$x\sqrt{x+y} = \sqrt{x^2(x+y)} \quad \text{when} \quad x, y \geq 0\,. \tag{9}$$

Let's look at the expression $\dfrac{x^2}{1+x^2}$. Can we simplify it?[1] Yes, by performing a virtual emplacement.

$$\frac{x^2}{1+x^2} = \frac{x^2+1-1}{1+x^2} = \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}\,. \tag{10}$$

Stuff like this comes up all the time in integration.

Now, as a real application, consider the integral:

$$\int \frac{x^2}{1+x^2}\,dx = \,? \tag{11}$$

Clearly, we can use the table integral (5), if we can message it into the correct form. But we can do that by employing the result of (10), as well, to get

$$\int \frac{x^2}{1+x^2}\,dx = \int 1\,dx - \int \frac{dx}{1+x^2}\,,$$

$$\text{hence} \quad \int \frac{x^2}{1+x^2}\,dx = x - \tan^{-1}x + C\,, \tag{12}$$

where C is an arbitrary constant.

4 The 'Carrington' (Differential) Equation vs. Integration by Parts

Integration by Parts is one of the most used techniques in the bag of gimmicks of indefinite integration. The technique provides us with an integration identity:

$$\int u\,dv = vu - \int v\,du\,. \tag{13}$$

The proof of it is based on the product rule of differentiation:

$$D_x[f(x)g(x)] = [D_xf(x)]g(x) + f(x)D_xg(x)\,. \tag{14}$$

Now, we integrate:

$$f(x)g(x) = \int g(x)D_xf(x)\,dx + \int f(x)D_xg(x)\,dx\,, \tag{15}$$

or,

$$f(x)g(x) = \int g(x)\,df + \int f(x)\,dg\,, \tag{16}$$

[1] By 'simplify' in this case, I mean to reduce the given expression to a sum of expressions, each of which is easier to integrate than the original expression.

Now, let $u = f(x)$ and $v = g(x)$, then this last equation becomes

$$uv = \int v\,du + \int u\,dv\,, \tag{17}$$

from which follows (13).

The Carrington[2] differential equation, associated to some integral, is any equation of the form[3]

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x) \tag{18}$$

that facilitates an integration problem. The next step is to integrate across (18).

This Carrington equation is not unique, though I will sometimes proffer to the reader 'the Carrington equation', which should be interpreted merely as the *particular* Carrington equation that I chose.

Heuristic: I often set $g(x)$ to be the integrand of the integral and then set $f(x) = x$.

Let's now do an example problem: Find the integral[4]

$$I = \int x \ln x\,dx\,, \tag{19}$$

We start with the Carrington differential equation (which follows the above heuristic):

$$D_x[x^2 \ln x] = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x\,. \tag{20}$$

Now we integrate:

$$\begin{aligned} x^2 \ln x &= 2I + \int x\,dx \\ &= 2I + \frac{1}{2}x^2\,. \end{aligned} \tag{21}$$

From this we get that

$$2I = x^2 \ln x - \frac{1}{2}x^2 + C'\,, \tag{22}$$

or

$$\int x \ln x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C\,. \tag{23}$$

5 The Integrals

Problem 1:

Find the integral

$$I = \int \sqrt{x^2 - x + 1}\,dx\,. \tag{24}$$

My solution will be to transform this integral into the form

$$\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2 + a^2}\right| + C\,. \tag{25}$$

[2]For the moment, this name is a place holder until I can come up with a better name for it. Dr. Carrington refers to a fictional character in the scifi movie *The Thing from Another World*, 1951. This equation is merely an ansatz and is, in most cases, not unique.

[3]The LHS of the Carrington Equation could contain a product of three or more factors, but that gets messy fast.

[4]I will often use the variables I, J, K, etc as placeholders for integrals to reduce the visual mess of the problem.

To accomplish this, I need to complete the square of the radicand $x^2 - x + 1$ by

$$x^2 - x + 1 = (x + A)^2 + B^2 = x^2 + 2Ax + A^2 + B^2 \,, \tag{26}$$

so that $A = -1/2$ and $B^2 = 3/4$. Therefore,

$$\begin{aligned} I = \int \sqrt{x^2 - x + 1}\, dx &= \int \sqrt{(x - 1/2)^2 + 3/4}\, d(x - 1/2) \\ &= \frac{x - 1/2}{2} \sqrt{(x - 1/2)^2 + 3/4} + \frac{3}{8} \log \left|(x - 1/2) + \sqrt{(x - 1/2)^2 + 3/4}\right| + C \,. \end{aligned} \tag{27}$$

Problem 2:

Find the integral

$$I = \int \ln \sqrt{1 + x^2}\, dx \,. \tag{28}$$

My first transformation is to this

$$I = \tfrac{1}{2} \int \ln(1 + x^2)\, dx \,. \tag{29}$$

I begin with the Carrington equation

$$D_x[x \ln(1 + x^2)] = \ln(1 + x^2) + \frac{2x^2}{1 + x^2} \,. \tag{30}$$

Next, we divide through by 1/2 and then integrate:

$$\tfrac{1}{2} x \ln(1 + x^2) = I + \int \frac{x^2}{1 + x^2}\, dx \,. \tag{31}$$

Now, we solve this for I and employ (12), to get

$$I = \tfrac{1}{2} x \ln(1 + x^2) - x + \tan^{-1} x + C \,. \tag{32}$$

And we're done.

Problem 3:

Find the integral

$$I = \int \frac{\tan x}{1 + \tan^2 x}\, dx \,. \tag{33}$$

But, $1 + \tan^2 x = \sec^2 x = \dfrac{1}{\cos^2 x}$ and $\tan x = \dfrac{\sin x}{\cos x}$, so that

$$I = \int \sin x \cos x\, dx \,. \tag{34}$$

At this point, I could resort to using another trig identity, but I'd prefer to do this more simply. Given that $\cos x\, dx = d(\sin x)$, then

$$I = \int \sin x\, d(\sin x) \,, \tag{35}$$

which has the form of

$$\int u\, du = \tfrac{1}{2} u^2 + C \,. \tag{36}$$

Thus,

$$I = \int \sin x \, d(\sin x) = \tfrac{1}{2} \sin^2 x + C \,. \tag{37}$$

Problem 4:

Find the integral

$$I = \int a^x \, dx \,. \tag{38}$$

Now, it may not be obvious, but a way to proceed could be to rewrite (38) in the form

$$\int e^{\alpha x} \, dx = \frac{1}{\alpha} e^{\alpha x} + C \,. \tag{39}$$

So, let's try this approach. Set

$$e^y = a^x \,, \tag{40}$$

and solve for y by taking the natural log of both sides:

$$y = x \ln a \,, \tag{41}$$

Thus,

$$I = \int e^{x \ln a} \, dx = \frac{1}{\ln a} e^{x \ln a} + C = \frac{1}{\ln a} a^x + C \,. \tag{42}$$

Problem 5:

Find the integral

$$I = \int \frac{9^x - 4^x}{3^x + 2^x} \, dx \,. \tag{43}$$

We will use the previous result, but to do so, we must first recognize that $9 = 3^2$ and $4 = 2^2$ and then factor the numerator by using the formula for the difference of two squares. Thus,

$$\begin{aligned} I &= \int \frac{3^{2x} - 2^{2x}}{3^x + 2^x} \, dx \\ &= \int \frac{(3^x - 2^x)(3^x + 2^x)}{3^x + 2^x} \, dx \\ &= \int 3^x - 2^x \, dx \\ &= \frac{1}{\ln 3} 3^x - \frac{1}{\ln 2} 2^x + C \,. \end{aligned} \tag{44}$$

Problem 6:

Find the integral

$$I = \int \ln x \, dx \,. \tag{45}$$

Proceeding by the standard approach, we have that

$$D_x[x \ln x] = \ln x + 1 \,. \tag{46}$$

So, on integrating

$$x \ln x = I + x - C \,. \tag{47}$$

Solving for I, we get

$$I = \int \ln x \, dx = x \ln x - x + C \,. \tag{48}$$

Problem 7:

Find the integral

$$I = \int x \ln \sqrt{1+x^2} \, dx \,. \tag{49}$$

My first transformation is to this

$$I = \tfrac{1}{2} \int x \ln(1+x^2) \, dx \,. \tag{50}$$

From here, using (48), it's pretty easy

$$\begin{aligned} I &= \tfrac{1}{4} \int (2x) \ln(1+x^2) \, dx \\ &= \tfrac{1}{4} \int \ln(1+x^2) \, d(1+x^2) \\ &= \tfrac{1}{4}(1+x^2)[\ln(1+x^2) - 1] + C \,. \end{aligned} \tag{51}$$

Problem 8:

Find the integral

$$I = \int x^2 \ln \sqrt{1+x^2} \, dx \,. \tag{52}$$

My first transformation is

$$I = \tfrac{1}{2} \int x^2 \ln(1+x^2) \, dx \,. \tag{53}$$

Then, I choose my Carrington equation as

$$\tfrac{1}{2}\tfrac{1}{3} D_x[x^3 \ln(1+x^2)] = \tfrac{1}{2} x^2 \ln(1+x^2) + \tfrac{1}{3} x^3 \frac{x}{1+x^2} \,. \tag{54}$$

Integrating, we get that

$$\tfrac{1}{2}\tfrac{1}{3}[x^3 \ln(1+x^2)] = I + J \,. \tag{55}$$

where

$$J \equiv \tfrac{1}{3} \int x^3 \frac{x}{1+x^2} \, dx \,. \tag{56}$$

So, once we do the integration for J, we can solve (55) for I. Now,

$$\begin{aligned} J &= \tfrac{1}{3} \int x^2 \frac{x^2}{1+x^2} \, dx \\ &= \tfrac{1}{3} \int (1+x^2) \frac{x^2}{1+x^2} \, dx - \tfrac{1}{3} \int \frac{x^2}{1+x^2} \, dx \quad \text{(using a virtual emplacement)} \\ &= \tfrac{1}{3} \int x^2 \, dx - \tfrac{1}{3} \int \frac{x^2}{1+x^2} \, dx \\ &= \tfrac{1}{9} x^3 - \tfrac{1}{3} x + \tan^{-1} x - C \quad \text{[using (12)]} \,. \end{aligned} \tag{57}$$

Solving for I, we get

$$I = \tfrac{1}{6}[x^3 \ln(1+x^2)] - \tfrac{1}{9} x^3 + \tfrac{1}{3} x - \tan^{-1} x + C \,. \tag{58}$$

Problem 9:

Find the integral

$$
\begin{aligned}
I &= \int \frac{dx}{x+a} \\
&= \int \frac{d(x+a)}{x+a} \\
&= \ln \big|x+a\big| + C\,.
\end{aligned} \tag{59}
$$

Problem 10:

Find the integral

$$
I = \int \frac{1}{x^2 - a^2}\, dx\,. \tag{60}
$$

The best way to proceed is to simplify the integrand by partial fractions.

$$
\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{\alpha}{(x-a)} + \frac{\beta}{(x+a)}\,, \tag{61}
$$

where α and β are real (sometimes complex) numbers to be determined by the usual means. On multiplying through by $x^2 - a^2$, we get

$$
1 = \alpha(x+a) + \beta(x-a)\,. \tag{62}
$$

The equation for the constants gives us

$$
1 = (\alpha - \beta)a\,. \tag{63}
$$

The equation for the first-order terms, gives us

$$
0 = \alpha + \beta\,. \tag{64}
$$

From which we get

$$
\alpha = \frac{1}{2a}\,, \qquad \beta = -\frac{1}{2a}\,. \tag{65}
$$

Hence,

$$
\begin{aligned}
I &= \int \frac{1}{x^2 - a^2}\, dx \\
&= \frac{1}{2a} \int \frac{1}{x-a}\, dx + \frac{1}{2a} \int \frac{1}{x+a}\, dx \\
&= \frac{1}{2a} \ln \big|x-a\big| - \frac{1}{2a} \ln \big|x+a\big| + C \\
&= \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right| + C\,.
\end{aligned} \tag{66}
$$

Problem 11:

Find the integral

$$
I = \int \frac{x^4}{1 - x^2}\, dx\,. \tag{67}
$$

Clearly, we can benefit from using the last integration. To that end, let

$$I = -\int \frac{x^4}{x^2-1}\,dx\,. \tag{68}$$

Next, we virtually emplace $1+(-1)$ in the numerator, and proceed as usual:

$$\begin{aligned} I &= -\int \frac{x^4-1+1}{x^2-1}\,dx \\ &= -\int \frac{x^4-1}{x^2-1} - \int \frac{1}{x^2-1}\,dx \\ &= -\int \frac{(x^2-1)(x^2+1)}{x^2-1} - \int \frac{1}{x^2-1}\,dx \\ &= -\int (x^2+1)\,dx - \tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C \\ &= -\tfrac{1}{3}x^3 - x - \tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C\,. \end{aligned} \tag{69}$$

Problem 12:

Find the integral

$$I = \int \tan x\,dx\,. \tag{70}$$

As a general rule, the way to approach integrands with trigonometric functions in them is either 1) to stay in the real numbers and employ one or more trig identities, or 2) replace the trig function by their complex exponential forms, when available. In this problem, we do the former.

$$\begin{aligned} I &= \int \tan x\,dx \\ &= \int \frac{\sin x}{\cos x}\,dx \\ &= \int \frac{-d(\cos x)}{\cos x} \\ &= -\ln\big|\cos x\big| + C\,. \end{aligned} \tag{71}$$

Problem 13:

Find the integral

$$I = \int \tan x \sec^2 x\,dx\,. \tag{72}$$

So, apply trig identities and then do the usual stuff

$$\begin{aligned} I &= \int \sin x \sec^3 x\,dx \\ &= \int \frac{-d(\cos x)}{\cos^3 x} \\ &= \tfrac{1}{2}\frac{1}{\cos^2 x} + C \\ &= \tfrac{1}{2}\sec^2 x + C\,. \end{aligned} \tag{73}$$

In like manner, one can show that (just differentiate through)

$$I = \int \tan x \sec x \, dx = \sec x + C \,. \tag{74}$$

Problem 14:

Easily, we show that

$$I = \int \sec^2 x \, dx = \tan x + C \,. \tag{75}$$

Problem 15:

Find the integral

$$I = \int \tan^2 x \, dx \,. \tag{76}$$

Use that $\tan^2 x + 1 = \sec^2 x$:

$$\begin{aligned} I &= \int \tan^2 x \, dx \\ &= \int (\sec^2 x - 1) \, dx \end{aligned}$$

$$\int \tan^2 x \, dx = \tan x - x + C \,. \tag{77}$$

Problem 16:

Find the integral

$$I = \int x \tan x \sec^2 x \, dx \,. \tag{78}$$

Using a clever Carrington equation:

$$D_x[x \tan^2 x] = \tan^2 x + 2x \tan x \sec^2 x \,. \tag{79}$$

On integrating, we get

$$x \tan^2 x = \int \tan^2 x \, dx + 2 \int x \tan x \sec^2 x \, dx \,, \tag{80}$$

or

$$x \tan^2 x = \tan x - x + 2I - C \,. \tag{81}$$

Therefore,

$$I = \tfrac{1}{2}(x \tan^2 x - \tan x + x) + C = \tfrac{1}{2}(x \sec^2 x - \tan x) + C \,. \tag{82}$$

Problem 17:

Given that

$$\int \sec x \, dx = \ln \Big| \sec x + \tan x \Big| + C \,, \tag{83}$$

find the integral

$$I = \int \tan^2 x \sec x \, dx \,. \tag{84}$$

This takes a little more effort. Let's start with a helper integral:

$$\begin{aligned} J &= \int \sec^3 x\, dx \\ &= \int (1+\tan^2 x)\sec x\, dx \\ &= \int \sec x\, dx + \int \tan^2 x \sec x\, dx \\ &= \ln\Big|\sec x + \tan x\Big| + I + C\,. \end{aligned} \tag{85}$$

Let's try this one other way:

$$D_x[\tan x \sec x] = \sec^3 x + \tan^2 x \sec x\,. \tag{86}$$

Now, integrate (and reverse sides of the equation):

$$J + I = \tan x \sec x + C'\,, \tag{87}$$

and from (85), we have

$$J - I = \ln\Big|\sec x + \tan x\Big| + C''\,. \tag{88}$$

From these we get

$$2J = \tan x \sec x + \ln\Big|\sec x + \tan x\Big| + C'''\,, \tag{89}$$

and from (85), we have

$$2I = \tan x \sec x - \ln\Big|\sec x + \tan x\Big| + C''''\,. \tag{90}$$

Simultaneously solving these, we get

$$I = \int \tan^2 x \sec x\, dx = \tfrac{1}{2}\tan x \sec x - \tfrac{1}{2}\ln\Big|\sec x + \tan x\Big| + C_1\,, \tag{91}$$

and from (85), we have

$$J = \int \sec^3 x\, dx = \tfrac{1}{2}\tan x \sec x + \tfrac{1}{2}\ln\Big|\sec x + \tan x\Big| + C_2\,. \tag{92}$$

Problem 18:

Given that

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\big(\frac{x}{a}\big)\,, \tag{93}$$

find the integral

$$I = \int \ln(a^2+x^2)\, dx\,. \tag{94}$$

Use the standard Carrington trick:

$$\begin{aligned} D_x[x\ln(a^2+x^2)] &= \ln(a^2+x^2) + x\frac{2x}{a^2+x^2} \\ &= \ln(a^2+x^2) + 2\frac{a^2+x^2-a^2}{a^2+x^2} \quad \text{(employ a virtual empl.)} \\ &= \ln(a^2+x^2) + 2\big(1 - \frac{a^2}{a^2+x^2}\big)\,. \end{aligned} \tag{95}$$

Integrating, we get

$$x \ln(a^2 + x^2) = I + 2x + 2a \tan^{-1}\big(\frac{x}{a}\big)\,. \tag{96}$$

Hence,

$$\int \ln(a^2 + x^2)\, dx = x \ln(a^2 + x^2) + 2x - 2a \tan^{-1}\big(\frac{x}{a}\big)\,. \tag{97}$$

Problem 19:

Find the integral

$$I(z) = \int \frac{\sqrt{z-1}}{z}\, dz\,, \tag{98}$$

where z is just a real variable.

A simple change of variables should do the trick this time. Let $w^2 = z - 1$ and thus $dz = 2wdw$. Then, our integral becomes

$$I(w) = 2 \int \frac{w^2}{w^2+1}\, dw\,. \tag{99}$$

Using (12), we get

$$I(w) = 2w - 2\tan^{-1} w + C\,. \tag{100}$$

Now, we change the variable back to z.

$$I(z) = 2\sqrt{z-1} - 2\tan^{-1}\sqrt{z-1} + C\,. \tag{101}$$

Problem 20:

Find the integral

$$I = \int \frac{dx}{\sqrt{1+x^2}}\,. \tag{102}$$

I think that the simplest way to proceed is with a trigonometric substitution. Let:

$$x = \tan u\,, \quad \text{then} \quad dx = \sec^2 u\, du\,. \tag{103}$$

Then

$$I(u) = \int \frac{\sec^2 u}{\sec u}\, du = \int \sec u\, du\,. \tag{104}$$

So, using (8), we get that

$$I(u) = \ln\big|\sec u + \tan u\big| + C\,. \tag{105}$$

Therefore,

$$I(x) = \ln\big|\sqrt{1+x^2} + x\big| + C\,. \tag{106}$$

Problem 21:

Find the integral

$$I = \int \frac{dx}{x^2\sqrt{1+x^2}}\,, \tag{107}$$

We start with the Carrington equation:

$$D_x[\frac{1}{x}\sqrt{1+x^2}] = -\frac{1}{x^2}\sqrt{1+x^2} + \frac{1}{\sqrt{1+x^2}}\,, \tag{108}$$

Now we integrate:

$$\frac{1}{x}\sqrt{1+x^2} = -I + \ln\left|\sqrt{1+x^2}+x\right| + C\,. \tag{109}$$

Therefore,

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \ln\left|\sqrt{1+x^2}+x\right| - \frac{1}{x}\sqrt{1+x^2} + C\,. \tag{110}$$

Problem 22:

Find the integral

$$I = \int \sqrt{1+x^2}\,dx\,. \tag{111}$$

No doubt there are many ways to approach this integral. But my purpose is to demonstrate the fullest variety of integration techniques before this series is over. Therefore, for this problem, I choose to do a variable substitution into the complex variables.

We start with the variable substitution:

$$x = i\sin u\,, \tag{112a}$$

$$dx = i\cos u\,du\,. \tag{112b}$$

Hence, we have that[5]

$$\begin{aligned} I(u) &= i\int \cos^2 u\,du \\ &= \frac{i}{2}\int (1+\cos 2u)\,du \\ &= \frac{i}{2}\big[u + \frac{\sin 2u}{2}\big] \\ &= \frac{i}{2}\big[u + \sin u\cos u\big] \end{aligned} \tag{113}$$

From (112a) we get that

$$\cos u = +\sqrt{1+x^2}\,. \tag{114}$$

Then, what about the variable u?

$$\sin u = \frac{e^{iu} - e^{-iu}}{2i}\,, \tag{115a}$$

$$\sinh iu = \frac{e^{iu} - e^{-iu}}{2}\,. \tag{115b}$$

Hence,

$$\sinh iu = i\sin u = x\,. \tag{116}$$

Therefore,

$$I(x) = \tfrac{1}{2}\big[\sinh^{-1}x + x\sqrt{1+x^2}\big] + C\,. \tag{117}$$

But wait! There's more!

Now, it has come to my attention that there are those who do not particularly like inverse hyperbolic functions. Therefore, we shall convert $\sinh^{-1}x$ to something more palatable.

[5] I will note list the specific trig identities, though I will make an effort to show my work on the complex-hyperbolic identities.

Let

$$y = \sinh^{-1} x\,. \tag{118}$$

Therefore,

$$x = \sinh y \equiv \frac{e^y - e^{-y}}{2}\,. \tag{119}$$

Now, let $z \equiv e^y$, then this last equation becomes

$$x = \sinh y \equiv \frac{z - z^{-1}}{2}\,. \tag{120}$$

With a little algebra, we get

$$z^2 - 2xz - 1 = 0\,, \tag{121}$$

with solution for z:

$$z = x + \sqrt{x^2+1}\,. \tag{122}$$

But $z \equiv e^y$, so

$$y = \ln\left|x + \sqrt{x^2+1}\right|\,. \tag{123}$$

Therefore,

$$I(x) = \tfrac{1}{2}\left[\,\ln\left|x + \sqrt{x^2+1}\right| + x\sqrt{1+x^2}\,\right]\,. \tag{124}$$

Problem 23:

Find the integral

$$I = \int \frac{dx}{(x^2-1)^{2/3}}\,dx\,, \tag{125}$$

Next, the Carrington equation:

$$\begin{aligned} D_x[x^2(x^2-1)^{1/3}] &= 2x(x^2-1)^{1/3} + x^2\tfrac{1}{3}(x^2-1)^{-2/3} \\ &= 2x(x^2-1)^{1/3} + x^2\tfrac{1}{3}\frac{x^2-1+1}{(x^2-1)^{2/3}} \\ &= 2x(x^2-1)^{1/3} + \tfrac{1}{3}(x^2-1)^{1/3} + \tfrac{1}{3}\frac{1}{(x^2-1)^{2/3}}\,, \end{aligned} \tag{126}$$

Now we integrate:

$$x^2(x^2-1)^{1/3} = 2\int x(x^2-1)^{1/3}dx + \tfrac{1}{3}\int (x^2-1)^{1/3}dx + \tfrac{1}{3}I\,. \tag{127}$$

The first term on the RHS integrates to

$$2\int x(x^2-1)^{1/3}dx = \int (x^2-1)^{1/3}d(x^2-1) = \tfrac{3}{4}(x^2-1)^{4/3}\,. \tag{128}$$

The second term on the RHS we calculated from the last problem:

$$\tfrac{1}{3}\int (x^2-1)^{1/3}dx = \tfrac{1}{3}[x(x^2-1)^{1/3} - \tfrac{1}{2}(x^2-1)^{1/3}]\,. \tag{129}$$

Therefore, the Carrington equation becomes:

$$x^2(x^2-1)^{1/3} = \tfrac{3}{4}(x^2-1)^{4/3} + \tfrac{1}{3}[x(x^2-1)^{1/3} - \tfrac{1}{2}(x^2-1)^{1/3}] + \tfrac{1}{3}I\,. \tag{130}$$

On multiplying through by 3 and then solving for I, we get

$$I = x^2(x^2-1)^{1/3} - \tfrac{9}{4}(x^2-1)^{4/3} - x(x^2-1)^{1/3} + \tfrac{3}{2}(x^2-1)^{1/3}\,. \tag{131}$$

Problem 24:

Find the integral

$$I = \int \sqrt{1-x^2}\, dx\,, \tag{132}$$

This time I want to use a trigonometric substitution. Let

$$x = \sin\theta\,, \tag{133}$$

then

$$\cos\theta = \sqrt{1-x^2}\,. \tag{134}$$

The relationships between θ and x is revealed in Figure 1.

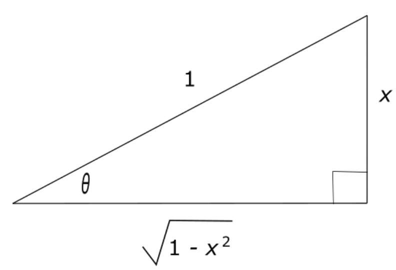

Figure 1. By choosing the hypotenuse to be unity, we simplify the trigonometric relationships given by the triangle.

Anyway,

$$\begin{aligned} I(\theta) &= \int \cos^2\theta\, d\theta \\ &= \tfrac{1}{2}\int (1+\cos 2\theta)\, d\theta \\ &= \tfrac{1}{2}\Big[\theta + \frac{\sin 2\theta}{2}\Big] \\ &= \tfrac{1}{2}\big[\theta + \sin\theta\cos\theta\big] + C\,. \end{aligned} \tag{135}$$

Then, back to the x-variable:

$$I(x) = \tfrac{1}{2}\big[\sin^{-1}x + x\sqrt{1-x^2}\,\big] + C\,. \tag{136}$$