

# Integration Techniques Paper 2

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## Abstract

Herein is a collection of integration techniques for indefinite integrals. I found the problems from a variety of sources, most of which I no longer know where they came from.

## 1 Introduction

Indefinite integrals, when they exist, result in functions, whereas, definite integrals, when they exist, result in numbers. The purpose of this paper is to present some techniques to perform indefinite integrals. My major strategy will be to reduce or convert a given integral into a form which is in a robust integral table, though, I may attempt to reduce the integral as far as I can take it. The integrals are presented in no particular order.

Warning: This paper assumes that the reader has a basic knowledge of integration (such as integration by parts), along with a familiarity with trigonometric and hyperbolic identities, logarithms, exponentials, partial fractions – generally speaking, the stuff found in a course on Algebra 2.

Note: The symbol  $D_x$  means to differentiate by  $x$ .

## 2 Reference Integrals and Derivatives for Later Use

First, the derivatives:

$$D_x \sin x = \cos x . \tag{1}$$

$$D_x \cos x = -\sin x . \tag{2}$$

$$D_x \sec x = \sec x \tan x . \tag{3}$$

$$D_x \tan x = \sec^2 x . \tag{4}$$

Now, the integrals:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C . \tag{5}$$

$$\int \ln x \, dx = \ln x(x - 1) + C . \tag{6}$$

$$\int \frac{du}{u} = \ln u + C . \tag{7}$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C . \tag{8}$$

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C . \tag{9}$$

$$\int \ln x \, dx = x \ln x - x + C. \quad (10)$$

$$\int x e^x \, dx = e^x (x - 1) + C. \quad (11)$$

$$\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C. \quad (12)$$

$$\int \frac{x^2}{1 + x^2} \, dx = x - \tan^{-1} x + C, \quad (13)$$

$$\int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec x| + C. \quad (14)$$

$$\int \ln(a^2 + x^2) \, dx = x \ln(a^2 + x^2) + 2x - 2a \tan^{-1} \left( \frac{x}{a} \right) + C. \quad (15)$$

$$\int \frac{dx}{\sqrt{1 + x^2}} = \ln |\sqrt{1 + x^2} + x| + C. \quad (16)$$

$$\int \sqrt{1 + x^2} \, dx = \frac{1}{2} [\sinh^{-1} x + x \sqrt{1 + x^2}] + C. \quad (17)$$

$$\int \sqrt{1 - x^2} \, dx = \frac{1}{2} [\sin^{-1} x + x \sqrt{1 - x^2}] + C. \quad (18)$$

$$\int \frac{x}{\sqrt{x^2 - a^2}} \, dx = \sqrt{x^2 - a^2} + C. \quad (19)$$

$$\int \frac{\sqrt{x^2 - a^2}}{x} \, dx = \sqrt{x^2 - a^2} - \cos^{-1} \frac{a}{x} + C. \quad (20)$$

### 3 Virtual Emplacement

So much of mathematics employs tricks that get used over and over in a wide variety of subject areas, yet go unnamed, and thus are hard to explain to one's readers when one uses them. Decades ago I invented the term *virtual emplacement* to refer to the algebraic action of adding a zero to an expression, or multiplying or dividing an expression by unity, or more generally, performing some function and its inverse to an expression, such as

$$x\sqrt{x+y} = \sqrt{x^2(x+y)} \quad \text{when } x, y \geq 0. \quad (21)$$

Let's look at the expression  $\frac{x^2}{1+x^2}$ . Can we simplify it?<sup>1</sup> Yes, by performing a virtual emplacement.

$$\frac{x^2}{1+x^2} = \frac{x^2 + 1 - 1}{1+x^2} = \frac{x^2 + 1}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}. \quad (22)$$

Stuff like this comes up all the time in integration.

Now, as a real application, consider the integral:

$$\int \frac{x^2}{1+x^2} \, dx = ? \quad (23)$$

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<sup>1</sup>By 'simplify' in this case, I mean to reduce the given expression to a sum of expressions, each of which is easier to integrate than the original expression.

Clearly, we can use the table integral (5), if we can message it into the correct form. But we can do that by employing the result of (22), as well, to get

$$\int \frac{x^2}{1+x^2} dx = \int 1 dx - \int \frac{dx}{1+x^2},$$

hence 
$$\int \frac{x^2}{1+x^2} dx = x - \tan^{-1} x + C, \tag{24}$$

where  $C$  is an arbitrary constant.

## 4 The ‘Carrington’ (Differential) Equation vs. Integration by Parts

Integration by Parts is one of the most used techniques in the bag of gimmicks of indefinite integration. The technique provides us with an integration identity:

$$\int u dv = vu - \int v du. \tag{25}$$

The proof of it is based on the product rule of differentiation:

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x). \tag{26}$$

Now, we integrate:

$$f(x)g(x) = \int g(x)D_x f(x) dx + \int f(x)D_x g(x) dx, \tag{27}$$

or,

$$f(x)g(x) = \int g(x) df + \int f(x) dg, \tag{28}$$

Now, let  $u = f(x)$  and  $v = g(x)$ , then this last equation becomes

$$uv = \int v du + \int u dv, \tag{29}$$

from which follows (25).

The Carrington<sup>2</sup> differential equation, associated to some integral, is any equation of the form<sup>3</sup>

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x) \tag{30}$$

that facilitates an integration problem. The next step is to integrate across (30).

This Carrington equation is not unique, though I will sometimes proffer to the reader ‘the Carrington equation’, which should be interpreted merely as the *particular* Carrington equation that I chose.

**Heuristic:** I often set  $g(x)$  to be the integrand of the integral and then set  $f(x) = x$ .

Let’s now do an example problem: Find the integral<sup>4</sup>

$$I = \int x \ln x dx, \tag{31}$$

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<sup>2</sup>For the moment, this name is a place holder until I can come up with a better name for it. Dr. Carrington refers to a fictional character in the scifi movie *The Thing from Another World*, 1951. This equation is merely an ansatz and is, in most cases, not unique.

<sup>3</sup>The LHS of the Carrington Equation could contain a product of three or more factors, but that gets messy fast.

<sup>4</sup>I will often use the variables  $I, J, K$ , etc as placeholders for integrals to reduce the visual mess of the problem.

We start with the Carrington differential equation (which follows the above heuristic):

$$D_x[x^2 \ln x] = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x. \quad (32)$$

Now we integrate:

$$\begin{aligned} x^2 \ln x &= 2I + \int x dx \\ &= 2I + \frac{1}{2}x^2. \end{aligned} \quad (33)$$

From this we get that

$$2I = x^2 \ln x - \frac{1}{2}x^2 + C', \quad (34)$$

or

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C. \quad (35)$$

## 5 The Integrals

### Problem 1:

Find the integral

$$I = \int x^2 \ln x dx. \quad (36)$$

I begin with the Carrington equation

$$D_x[x^3 \ln x] = 3x^2 \ln x + x^2. \quad (37)$$

Next, we integrate:

$$x^3 \ln x = 3I + \frac{1}{3}x^3, \quad (38)$$

yielding

$$\int x^2 \ln x dx = \frac{1}{3}x^3 \left[ \ln x - \frac{1}{3} \right] + C. \quad (39)$$

### Problem 2:

Given that

$$D_u e^u = e^u, \quad (40)$$

show that

$$\int x e^x dx = e^x (x - 1) + C. \quad (41)$$

Proof: Start with a Carrington equation:

$$D_x[xe^x] = e^x + xe^x. \quad (42)$$

Now, integrate:

$$xe^x = e^x + I - C. \quad (43)$$

On rearranging this, we get (41).

**Problem 3:**

Find the integral

$$I = \int x^2 e^x dx. \quad (44)$$

Start with a Carrington equation:

$$D_x[x^2 e^x] = 2x e^x + x^2 e^x. \quad (45)$$

Now, integrate:

$$x^2 e^x = 2 \int x e^x dx + I - C. \quad (46)$$

On rearranging this and using (41), we get

$$I = x^2 e^x - 2e^x(x - 1) + C = e^x(x^2 - 2x + 2) + C. \quad (47)$$

**Problem 4:**

Find the integral

$$I = \int x^3 e^x dx. \quad (48)$$

Start with a Carrington equation:

$$D_x[x^3 e^x] = 3x^2 e^x + x^3 e^x. \quad (49)$$

Now, integrate:

$$x^3 e^x = 3 \int x^2 e^x dx + I - C. \quad (50)$$

On rearranging this and using (47), we get

$$I = e^x(x^3 - 3x^2 + 6x - 6) + C. \quad (51)$$

**Problem 5:**

Given that

$$D_u e^{-u} = -e^{-u}, \quad (52)$$

show that

$$\int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + C. \quad (53)$$

Proof:

$$\begin{aligned} \int x e^{-x^2} dx &= \frac{1}{2} \int e^{-x^2} d(x^2) \\ &= \frac{1}{2} \int e^{-u} d(u) \quad \text{let } u = x^2 \\ &= -\frac{1}{2} e^{-u} + C \\ &= -\frac{1}{2} e^{-x^2} + C. \end{aligned} \quad (54)$$

**Problem 6:**

Using the result of the last problem, show that

$$\int x^3 e^{-x^2} dx = -\frac{1}{2} e^{-x^2} (x^2 + 1) + C. \quad (55)$$

Proof:

$$\begin{aligned} \int x^3 e^{-x^2} dx &= \frac{1}{2} \int (-x^2) e^{-x^2} d(-x^2) \\ &= \frac{1}{2} \int u e^u d(u) \quad (\text{let } u = -x^2) \\ &= \frac{1}{2} e^u (u - 1) + C \\ &= -\frac{1}{2} e^{-x^2} (x^2 + 1) + C. \end{aligned} \quad (56)$$

**Problem 7:**

Find the integral

$$I = \int \frac{6x + 7}{(x + 2)^2} dx. \quad (57)$$

One way to proceed is to simplify the integrand by partial fractions.<sup>5</sup>

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}, \quad (58)$$

where  $A$  and  $B$  are real (sometimes complex) numbers to be determined by the usual means, which gives us

$$A = 6 \quad \text{and} \quad B = -5. \quad (59)$$

Hence,

$$\begin{aligned} I &= \int \frac{6}{x + 2} dx + \int \frac{-5}{(x + 2)^2} dx \\ &= 6 \ln |x + 2| + 5 \frac{1}{x + 2} + C. \end{aligned} \quad (60)$$

**Problem 8:**

Find the integral

$$I = \int \frac{dx}{(1 + x^2)^2} dx. \quad (61)$$

Let's try a Carrington equation:

$$D_x \left[ x \frac{1}{1 + x^2} \right] = \frac{1}{1 + x^2} - \frac{2x^2}{(1 + x^2)^2}, \quad (62)$$

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<sup>5</sup>The method of partial fractions can be found in algebra books and on line, no doubt.

Integration gives us

$$\begin{aligned}
 \frac{x}{1+x^2} &= \int \frac{1}{1+x^2} dx - \int \frac{2x^2}{(1+x^2)^2} dx \\
 &= \tan^{-1} x - 2 \int \frac{(1+x^2) - 1}{(1+x^2)^2} dx \\
 &= \tan^{-1} x - 2 \int \frac{1}{(1+x^2)} dx + 2 \int \frac{1}{(1+x^2)^2} dx \\
 &= -\tan^{-1} x + 2I - C'.
 \end{aligned} \tag{63}$$

Hence,

$$I = \frac{1}{2} \frac{x}{1+x^2} + \frac{1}{2} \tan^{-1} x + C. \tag{64}$$

**Problem 9:**

Find the integral

$$I = \int \frac{\sqrt{a^2 - x^2}}{x} dx. \tag{65}$$

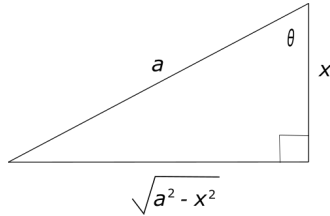


Figure 1. By choosing the hypotenuse to be  $a$ , we simplify the trigonometric relationships given by the triangle.

This time I want to use a trigonometric substitution. Let

$$\cos \theta = \frac{x}{a}, \tag{66}$$

then

$$\sin \theta = \frac{\sqrt{a^2 - x^2}}{a} \quad \text{and} \quad \tan \theta = \frac{\sqrt{a^2 - x^2}}{x}, \tag{67}$$

and

$$dx = -a \sin \theta d\theta. \tag{68}$$

Substituting into (65), we get

$$\begin{aligned}
 I(\theta) &= -a \int \frac{\sin^2 \theta}{\cos \theta} d\theta \\
 &= -a \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta \\
 &= -a \left[ \int \frac{1}{\cos \theta} d\theta - \int \cos \theta d\theta \right] \\
 &= -a \left[ \int \sec \theta d\theta - \int \cos \theta d\theta \right] \\
 &= -a \left[ \ln |\sec \theta + \tan \theta| - \sin \theta \right] + C.
 \end{aligned} \tag{69}$$

Then, back to the  $x$ -variable:

$$I(x) = -a \left[ \ln \left| \frac{a}{x} + \frac{\sqrt{a^2 - x^2}}{x} \right| - \frac{\sqrt{a^2 - x^2}}{a} \right] + C'. \quad (70)$$

And finally,

$$\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + C. \quad (71)$$

**Problem 10:**

Find the integral

$$J = \int x \cos x dx. \quad (72)$$

One way to proceed is to recast the problem into the complex numbers.

$$I = \int x e^{ix} dx, \quad (73)$$

where

$$e^{ix} = \cos x + i \sin x. \quad (74)$$

I want to apply

$$\int u e^u du = e^u (u - 1) + C \quad (75)$$

to Equation (73), with  $u = ix$ . So,

$$I = - \int (ix) e^{ix} d(ix). \quad (76)$$

Thus,

$$\begin{aligned} I(u) &= - \int u e^u du \\ &= -e^u (u - 1) + C \\ I(x) &= -e^{ix} (ix - 1) + C \\ &= -(\cos x + i \sin x)(ix - 1) + C \\ &= (\cos x + x \sin x + C_1) + i(\sin x - x \cos x + C_2), \end{aligned} \quad (77)$$

where  $C_1$  and  $C_2$  are, respectively, the real and imaginary parts of  $C$ .

However,  $\int x e^{ix} dx$  can be expanded into its real and imaginary components prior to integrating, yielding

$$\int x e^{ix} dx = \int x \cos x dx + i \int x \sin x dx, \quad (78)$$

So, we simply equate the real and imaginary parts of the last two equations, to get

$$\begin{aligned} \int x \cos x dx &= \cos x + x \sin x + C_1, \\ \int x \sin x dx &= \sin x - x \cos x + C_2. \end{aligned} \quad (79)$$

**Problem 11:**

Find the integral

$$I = \int x^2 \sin x \, dx. \quad (80)$$

We begin with the Carrington equation:

$$D_x[x^2 \cos x] = 2x \cos x - x^2 \sin x. \quad (81)$$

And now integrate:

$$x^2 \cos x = 2 \int x \cos x \, dx - I. \quad (82)$$

Using the result of the last problem, we get

$$I = -x^2 \cos x + 2(\cos x + x \sin x) + C. \quad (83)$$

**Problem 12:**

Find the integral

$$I = \int \frac{1}{1 + e^x} \, dx. \quad (84)$$

A simple virtual emplacement should to it:

$$\begin{aligned} \int \frac{1}{1 + e^x} \, dx &= \int \frac{1 + e^x - e^x}{1 + e^x} \, dx \\ &= \int \frac{1 + e^x}{1 + e^x} \, dx - \int \frac{e^x}{1 + e^x} \, dx \\ &= \int dx - \int \frac{d(1 + e^x)}{1 + e^x} \\ &= x - \ln(1 + e^x) + C. \end{aligned} \quad (85)$$

**Problem 13:**

Find the integral

$$I = \int \frac{1 - e^{-2x}}{1 + e^{-4x}} \, dx. \quad (86)$$

First, multiply numerator and denominator through by  $e^{4x}$  and then follow-up with a virtual emplacement:

$$\begin{aligned} \int \frac{1 - e^{-2x}}{1 + e^{-4x}} \, dx &= \int \frac{e^{4x} - e^{2x}}{e^{4x} + 1} \, dx \\ &= \int \frac{e^{4x}}{e^{4x} + 1} \, dx - \int \frac{e^{2x}}{e^{4x} + 1} \, dx \\ &= \frac{1}{4} \int \frac{d(e^{4x} + 1)}{e^{4x} + 1} - \frac{1}{2} \int \frac{d(e^{2x})}{(e^{2x})^2 + 1} \\ &= \frac{1}{4} \ln(e^{4x} + 1) - \frac{1}{2} \tan^{-1}(e^{2x}) + C. \end{aligned} \quad (87)$$

Sometimes, a pair of integrals is more easily solved simultaneously!

**Problem 14:**

Find the integrals

$$I = \int e^x \sin x \, dx, \quad (88a)$$

$$J = \int e^x \cos x \, dx. \quad (88b)$$

Next, we need a pair of Carringtons:

$$D_x[e^x \sin x] = e^x \sin x + e^x \cos x, \quad (89a)$$

$$D_x[e^x \cos x] = e^x \cos x - e^x \sin x. \quad (89b)$$

Now, we integrate the pair:

$$e^x \sin x = I + J, \quad (90a)$$

$$e^x \cos x = J - I. \quad (90b)$$

On solving for  $I$  and  $J$ , simultaneously:

$$I = \frac{1}{2}e^x(\sin x - \cos x), \quad (91a)$$

$$J = \frac{1}{2}e^x(\sin x + \cos x). \quad (91b)$$

**Problem 15:**

Find the integral

$$I = \int \cos^2 x \, dx. \quad (92)$$

Next, a Carrington equation:

$$D_x[\sin x \cos x] = \cos^2 x - \sin^2 x. \quad (93)$$

But

$$\cos^2 x - \sin^2 x = 2 \cos^2 x - 1, \quad (94)$$

So,

$$D_x[\sin x \cos x] = 2 \cos^2 x - 1. \quad (95)$$

Now, integrate

$$\sin x \cos x = 2I - x + C'. \quad (96)$$

From which we get that

$$I = \frac{1}{2}(\sin x \cos x + x) + C. \quad (97)$$

Question: Where do strange integrands come from? Ans: Sometimes from trying to integrate reasonable integrands.

**Problem 16:**

Find the integral

$$I = \int \frac{1}{1+x^5} dx. \quad (98)$$

Try the Carrington equation:

$$\begin{aligned} D_x[x \ln(1+x^5)] &= \ln(1+x^5) + 5 \frac{x^5}{1+x^5} \\ &= \ln(1+x^5) + 5 \frac{1+x^5-1}{1+x^5} \\ &= \ln(1+x^5) + 5 \frac{1+x^5}{1+x^5} - 5 \frac{1}{1+x^5} \\ &= \ln(1+x^5) + 5 - 5 \frac{1}{1+x^5}, \end{aligned} \quad (99)$$

Now, we integrate:

$$x \ln(1+x^5) = J + 5x - 5I, \quad (100)$$

where

$$J = \int \ln(1+x^5) dx. \quad (101)$$

Thus, the strange integral  $J$  arose in this particular effort to find  $I$ . So, if we can find either  $I$  or  $J$ , we can solve for the other integral.

There is a straightforward way to proceed to do the integral in (98), and that is to fully factor  $1+x^5$  and then use the method of partial fractions.

**Problem 17:**

Find the integral

$$I = \int \frac{8x^3 dx}{4x^2 + 4x + 5}. \quad (102)$$

Most of the heavy lifting in this problem will be by virtual emplacements.<sup>6</sup>

$$\begin{aligned} I &= \int \frac{8x^3 + (8x^2 + 10x) - (8x^2 + 10x)}{4x^2 + 4x + 5} dx \\ &= \int \frac{8x^3 + 8x^2 + 10x}{4x^2 + 4x + 5} dx - \int \frac{8x^2 + 10x}{4x^2 + 4x + 5} dx \\ &= \int 2x dx - 2 \int \frac{4x^2 + 5x}{4x^2 + 4x + 5} dx \\ &= x^2 - 2 \int \frac{4x^2 + 4x + 5}{4x^2 + 4x + 5} dx - 2 \int \frac{x-5}{4x^2 + 4x + 5} dx \\ &= x^2 - 2 \int dx - 2 \int \frac{x-5}{4x^2 + 4x + 5} dx \\ &= x^2 - 2x - 2 \int \frac{x-5}{4x^2 + 4x + 5} dx. \end{aligned} \quad (103)$$

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<sup>6</sup>One could also approach this integrand by using long division.

According to Wolframalpha.com:

$$\int \frac{x-5}{4x^2+4x+5} dx = \frac{1}{8}[-11 \tan^{-1}(1/2+x) + \ln(5+4x+4x^2)]. \quad (104)$$

Therefore,

$$\int \frac{8x^3 dx}{4x^2+4x+5} = x^2 - 2x - \frac{1}{4}[-11 \tan^{-1}(1/2+x) + \ln(5+4x+4x^2)] + C. \quad (105)$$

**Problem 18:**

Given that

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad (106)$$

find the integral

$$I = \int \sin^{-1} x dx. \quad (107)$$

We'll try the Carrington equation

$$D_x[x \sin^{-1} x] = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}. \quad (108)$$

On integration, we get

$$x \sin^{-1} x = I + \int \frac{x}{\sqrt{1-x^2}} dx. \quad (109)$$

But

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} = -\sqrt{1-x^2}. \quad (110)$$

Therefore,

$$I = x \sin^{-1} x + \sqrt{1-x^2} + C. \quad (111)$$

**Problem 19:**

Find the integral

$$I = \int \sin(ax) \cos(bx) dx \quad (|a| \neq |b|). \quad (112)$$

We'll try the Carrington equation

$$D_x[\sin(ax) \sin(bx)] = a \cos(ax) \sin(bx)x + b \sin(ax) \cos(bx). \quad (113)$$

Now, we integrate, to get

$$\sin(ax) \sin(bx)x = aJ + bI. \quad (114)$$

where

$$J \equiv \int \cos(ax) \sin(bx) dx. \quad (115)$$

We'll try the Carrington equation

$$D_x[\cos(ax) \cos(bx)] = -a \sin(ax) \cos(bx) - b \cos(ax) \sin(bx). \quad (116)$$

Now, we integrate, to get

$$\cos(ax) \cos(bx) = -aI - bJ. \quad (117)$$

On solving (114) and (117) simultaneously for  $I$  and  $J$ , we get

$$\begin{aligned} I &= -\frac{b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)}{a^2 - b^2} + C_1 \\ J &= \frac{a \sin(ax) \sin(bx) + b \cos(ax) \cos(bx)}{a^2 - b^2} + C_2. \end{aligned} \quad (118)$$

**Problem 20:**

Find the integral

$$I = \int \frac{dz}{\sqrt{z^2 - 1}} \quad (z > 1). \quad (119)$$

Given that

$$\cosh^2 u - 1 = \sinh^2 u, \quad (120)$$

let

$$\begin{aligned} z &= \cosh u, \\ z^2 - 1 &= \sinh^2 u, \\ dz &= \sinh u \, du. \end{aligned} \quad (121)$$

Hence

$$I(u) = \int \frac{\sinh u \, du}{\sinh u} = u + C. \quad (122)$$

Therefore,

$$I(z) = \cosh^{-1} z + C. \quad (123)$$

But, then again, some people don't care for inverse hyperbolic functions, so we need some other way to express  $u$  as a function of  $z$ . Since

$$z = \cosh u \equiv \frac{e^u + e^{-u}}{2}, \quad (124)$$

rewrite this last equation, using  $y = e^u$ :

$$2z = y + y^{-1}. \quad (125)$$

Now, multiply through by  $y$  and express the result as a quadratic equation in the variable  $y$ :

$$y^2 - 2zy + 1 = 0, \quad (126)$$

which has solution

$$y = z + \sqrt{z^2 - 1}. \quad (127)$$

Therefore,

$$u = \ln(z + \sqrt{z^2 - 1}). \quad (128)$$

And finally,

$$\int \frac{dz}{\sqrt{z^2 - 1}} = \ln(z + \sqrt{z^2 - 1}) + C. \quad (129)$$

**Problem 21:**

Find the integral

$$I = \int \ln(z + \sqrt{z^2 - 1}) dz. \quad (130)$$

Try the Carrington

$$D_z[z \ln(z + \sqrt{z^2 - 1})] = \ln(z + \sqrt{z^2 - 1}) + z \frac{1 + \frac{z}{\sqrt{z^2 - 1}}}{z + \sqrt{z^2 - 1}}. \quad (131)$$

let

$$T(z) = \frac{1 + \frac{z}{\sqrt{z^2 - 1}}}{z + \sqrt{z^2 - 1}}. \quad (132)$$

It can be shown that  $T(z)$  reduces to  $1/\sqrt{z^2 - 1}$ . (Take as your first step to multiply both numerator and denominator by  $z - \sqrt{z^2 - 1}$ .)

So, on integrating (131), we get

$$z \ln(z + \sqrt{z^2 - 1}) = I + \int \frac{z}{\sqrt{z^2 - 1}} dz. \quad (133)$$

On solving for  $I$  and using (19), we have

$$I = z \ln(z + \sqrt{z^2 - 1}) - \sqrt{z^2 - 1} + C. \quad (134)$$