Integration Techniques Paper 3

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Abstract

Herein is a collection of integration techniques for indefinite integrals. I found the problems from a variety of sources, most of which I no longer know where they came from.

1 Introduction

Indefinite integrals, when they exist, result in functions, whereas, definite integrals, when they exist, result in numbers. The purpose of this paper is to present some techniques to perform indefinite integrals. My major strategy will be to reduce or convert a given integral into a form which is in a robust integral table, though, I may attempt to reduce the integral as far as I can take it. The integrals are presented in no particular order.

Warning: This paper assumes that the reader has a basic knowledge of integration (such as integration by parts), along with a familiarity with trigonometric and hyperbolic identities, logarithms, exponentials, partial fractions – generally speaking, the stuff found in a course on Algebra 2.

Note: The symbol D_x means to differentiate by x.

2 Table of Integrals and Derivatives for Later Use

First, the derivatives:

$$D_x \sin x = \cos x \,. \tag{1}$$

$$D_x \cos x = -\sin x \,. \tag{2}$$

$$D_x \sec x = \sec x \tan x \,. \tag{3}$$

$$D_x \tan x = \sec^2 x \,. \tag{4}$$

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \,, \tag{5}$$

Some hyperbolic trigonometric functions and derivatives:

$$\cosh^2 x - \sinh^2 x = 1 \,. \tag{6}$$

$$D_x \cosh x = \sinh x \,. \tag{7}$$

$$D_x \sinh x = \cosh x \,. \tag{8}$$

Now, the integrals:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \,. \tag{9}$$

$$\int \frac{du}{u} = \ln u + C\,. \tag{10}$$

$$\int \sec x\, dx = \ln\Big|\sec x + \tan x\Big| + C\,. \tag{11}$$

$$\int \csc^2 x\, dx = -\cot x + C\,. \tag{12}$$

$$\int a^x\, dx = \frac{1}{\ln a} a^x + C\,. \tag{13}$$

$$\int \ln x\, dx = x \ln x - x + C\,. \tag{14}$$

$$\int x e^x\, dx = e^x (x-1) + C\,. \tag{15}$$

$$\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \ln\Big|\frac{x-a}{x+a}\Big| + C\,. \tag{16}$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C\,. \tag{17}$$

$$\int \frac{1}{x\sqrt{a^2 - x^2}}\, dx = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x} + C\,. \tag{18}$$

$$\int \frac{x^2}{1 + x^2}\, dx = x - \tan^{-1} x + C\,, \tag{19}$$

$$\int \tan x\, dx = -\ln\big|\cos x\big| + C = \ln\big|\sec x\big| + C\,. \tag{20}$$

$$\int \ln(a^2 + x^2)\, dx = x \ln(a^2 + x^2) + 2x - 2a \tan^{-1}\big(\frac{x}{a}\big) + C\,. \tag{21}$$

$$\int \frac{dx}{\sqrt{1 + x^2}} = \ln\big|\sqrt{1 + x^2} + x\big| + C\,. \tag{22}$$

$$\int \sqrt{1 + x^2}\, dx = \tfrac{1}{2}\big[\sinh^{-1} x + x\sqrt{1 + x^2}\big] + C\,. \tag{23}$$

$$\int \sqrt{1 - x^2}\, dx = \tfrac{1}{2}\big[\sin^{-1} x + x\sqrt{1 - x^2}\,\big] + C\,. \tag{24}$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C\,. \tag{25}$$

$$\int \frac{x}{\sqrt{x^2 - a^2}}\, dx = \sqrt{x^2 - a^2} + C\,. \tag{26}$$

$$\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - \cos^{-1} \frac{a}{x} + C\,. \tag{27}$$

$$\int x^2 \ln x\, dx = \tfrac{1}{3} x^3 \big[\ln x - \tfrac{1}{3}\,\big] + C\,. \tag{28}$$

$$\int xe^x\,dx = e^x(x-1) + C\,. \tag{29}$$

$$\int x^2e^x\,dx = e^x(x^2-2x+2) + C\,. \tag{30}$$

$$\int x^3e^x\,dx = e^x(x^3-3x^2+6x-6) + C\,. \tag{31}$$

$$\int x^3e^{-x^2}\,dx = -\tfrac{1}{2}e^{-x^2}(x^2+1) + C\,. \tag{32}$$

$$\int \frac{dx}{(1+x^2)^2}\,dx = \tfrac{1}{2}\frac{x}{1+x^2} + \tfrac{1}{2}\tan^{-1}x + C\,. \tag{33}$$

$$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C\,. \tag{34}$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x+\sqrt{x^2+a^2}\right| + C\,. \tag{35}$$

$$\int x\cos x\,dx = \cos x + x\sin x + C_1\,, \tag{36a}$$

$$\int x\sin x\,dx = \sin x - x\cos x + C_2\,. \tag{36b}$$

$$\int x^2\sin\,x\,dx = -x^2\cos\,x + 2(\cos x + x\sin x) + C\,. \tag{37}$$

$$\int e^x\sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x)\,, \tag{38a}$$

$$\int e^x\cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x)\,. \tag{38b}$$

$$\int \sin^{-1}x\,dx = x\,\sin^{-1}x + \sqrt{1-x^2} + C\,. \tag{39}$$

$$\int \sin(ax)\cos(bx)\,dx = -\frac{b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)}{a^2-b^2} + C_1 \tag{40}$$

$$\int \cos(ax)\sin(bx)\,dx = \frac{a\sin(ax)\sin(bx) + b\cos(ax)\cos(bx)}{a^2-b^2} + C_2\,. \tag{41}$$

$$\int \frac{dz}{\sqrt{z^2-1}} = \cosh^{-1}z + C = \ln\,(z+\sqrt{z^2-1}) + C \quad (z>1)\,. \tag{42}$$

$$\int \ln\,(z+\sqrt{z^2-1})\,dz = z\ln\,(z+\sqrt{z^2-1}) - \sqrt{z^2-1} + C\,. \tag{43}$$

3 Virtual Emplacement

So much of mathematics employs tricks that get used over and over in a wide variety of subject areas, yet go unnamed, and thus are hard to explain to one's readers when one uses them. Decades ago I invented the term *virtual emplacement* to refer to the algebraic action of adding a zero to an expression, or multiplying or dividing an expression by unity, or more generally, performing some function and its inverse to an expression, such as

$$x\sqrt{x+y} = \sqrt{x^2(x+y)} \quad \text{when} \quad x, y \geq 0 \,. \tag{44}$$

Let's look at the expression $\dfrac{x^2}{1+x^2}$. Can we simplify it?[1] Yes, by performing a virtual emplacement.

$$\frac{x^2}{1+x^2} = \frac{x^2+1-1}{1+x^2} = \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2} \,. \tag{45}$$

Stuff like this comes up all the time in integration.

Now, as a real application, consider the integral:

$$\int \frac{x^2}{1+x^2}\, dx = \,? \tag{46}$$

Clearly, we can use the table integral (9), if we can message it into the correct form. But we can do that by employing the result of (45), as well, to get

$$\int \frac{x^2}{1+x^2}\, dx = \int 1\, dx - \int \frac{dx}{1+x^2} \,,$$

$$\text{hence} \quad \int \frac{x^2}{1+x^2}\, dx = x - \tan^{-1} x + C \,, \tag{47}$$

where C is an arbitrary constant.

4 The 'Carrington' (Differential) Equation vs. Integration by Parts

Integration by Parts is one of the most used techniques in the bag of gimmicks of indefinite integration. The technique provides us with an integration identity:

$$\int u\, dv = vu - \int v\, du \,. \tag{48}$$

The proof of it is based on the product rule of differentiation:

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x) \,. \tag{49}$$

Now, we integrate:

$$f(x)g(x) = \int g(x) D_x f(x)\, dx + \int f(x) D_x g(x)\, dx \,, \tag{50}$$

or,

$$f(x)g(x) = \int g(x)\, df + \int f(x)\, dg \,, \tag{51}$$

[1] By 'simplify' in this case, I mean to reduce the given expression to a sum of expressions, each of which is easier to integrate than the original expression.

Now, let $u = f(x)$ and $v = g(x)$, then this last equation becomes

$$uv = \int v\,du + \int u\,dv\,, \tag{52}$$

from which follows (48).

The Carrington[2] differential equation, associated to some integral, is any equation of the form[3]

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x) \tag{53}$$

that facilitates an integration problem. The next step is to integrate across (53).

This Carrington equation is not unique, though I will sometimes proffer to the reader 'the Carrington equation', which should be interpreted merely as the *particular* Carrington equation that I chose.

Heuristic: I often set $g(x)$ to be the integrand of the integral and then set $f(x) = x$.

Let's now do an example problem: Find the integral[4]

$$I = \int x \ln x\,dx\,, \tag{54}$$

We start with the Carrington differential equation (which follows the above heuristic):

$$D_x[x^2 \ln x] = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x\,. \tag{55}$$

Now we integrate:

$$\begin{aligned} x^2 \ln x &= 2I + \int x\,dx \\ &= 2I + \frac{1}{2}x^2\,. \end{aligned} \tag{56}$$

From this we get that

$$2I = x^2 \ln x - \frac{1}{2}x^2 + C'\,, \tag{57}$$

or

$$\int x \ln x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C\,. \tag{58}$$

[2]For the moment, this name is a place holder until I can come up with a better name for it. Dr. Carrington refers to a fictional character in the scifi movie *The Thing from Another World*, 1951. This equation is merely an ansatz and is, in most cases, not unique.

[3]The LHS of the Carrington Equation could contain a product of three or more factors, but that gets messy fast.

[4]I will often use the variables I, J, K, etc as placeholders for integrals to reduce the visual mess of the problem.

5 The Integrals

Problem 1:

Find the integral

$$I = \int \frac{1}{1+\cos x}\,dx\,. \tag{59}$$

The way to get a better denominator is to multiply both numerator and denominator by $1-\cos x$ and proceed:

$$\begin{aligned}
\int \frac{1}{1+\cos x}\,dx &= \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}\,dx \\
&= \int \frac{1-\cos x}{(1-\cos^2 x)}\,dx \\
&= \int \frac{1-\cos x}{\sin^2 x}\,dx \\
&= \int \frac{1}{\sin^2 x}\,dx - \int \frac{\cos x}{\sin^2 x}\,dx \\
&= \int \csc^2 x\,dx - \int \frac{d(\sin x)}{\sin^2 x}
\end{aligned}$$

$$. \tag{60}$$

Using (12), we get that

$$\int \frac{1}{1+\cos x}\,dx = -\cot x + \csc x + C\,. \tag{61}$$

Problem 2:

Find the integral

$$I = \int \frac{1}{\sqrt{a^2-x^2}}\,dx\,. \tag{62}$$

Let's start with the Carrington equation:

$$D_x[x\sqrt{a^2-x^2}] = \sqrt{a^2-x^2} + \frac{-x^2}{\sqrt{a^2-x^2}}\,. \tag{63}$$

Integrating this gives

$$\begin{aligned}
x\sqrt{a^2-x^2} &= \int \sqrt{a^2-x^2}\,dx + \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx \\
&= \int \sqrt{a^2-x^2}\,dx + \int \frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx \\
&= \int \sqrt{a^2-x^2}\,dx + \int \frac{(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx + \int \frac{-a^2}{\sqrt{a^2-x^2}}\,dx \\
&= 2\int \sqrt{a^2-x^2}\,dx - a^2 I\,.
\end{aligned} \tag{64}$$

Hence,

$$I = \frac{1}{a^2}\Big[\,2\int \sqrt{a^2-x^2}\,dx - x\sqrt{a^2-x^2}\,\Big]\,. \tag{65}$$

On using (25), we get

$$I = \sin^{-1}\frac{x}{a} + C\,. \tag{66}$$

Problem 3:

Find the integral

$$I = \int \frac{\sqrt{9-x^2}}{x^2}\,dx\,. \tag{67}$$

Let's start with the Carrington equation:

$$D_x[\frac{1}{x}\sqrt{9-x^2}] = -\frac{1}{x^2}\sqrt{9-x^2} + \frac{1}{x}\frac{-x}{\sqrt{9-x^2}}\,. \tag{68}$$

Integrating this gives

$$\frac{1}{x}\sqrt{9-x^2} = -I - \int \frac{1}{\sqrt{9-x^2}}\,dx\,. \tag{69}$$

Solving for I, we have

$$I = -\frac{1}{x}\sqrt{9-x^2} - \int \frac{1}{\sqrt{9-x^2}}\,dx\,. \tag{70}$$

Using the result of the last problem, we have that

$$I = -\frac{1}{x}\sqrt{9-x^2} - \sin^{-1}\frac{x}{3} + C\,. \tag{71}$$

Problem 4:

Find the integral

$$I = \int \frac{dx}{x\sqrt{1-x^2}}\,. \tag{72}$$

Let's start with a variable substitution:

$$x = \cos u \quad \text{then} \quad dx = -\sin u\,du \quad \text{and} \quad \sin u = \sqrt{1-x^2}\,. \tag{73}$$

Substituting in, we get

$$I = -\int \sec u\,du = -\ln\Big|\sec u + \tan u\Big| + C\,. \tag{74}$$

Thus

$$I = -\ln\Big|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\Big| + C\,, \tag{75}$$

which simplifies to

$$I = -\ln\Big|\frac{1+\sqrt{1-x^2}}{x}\Big| + C\,. \tag{76}$$

Lemma 1

$$D_x \tan^{-1} x = \frac{1}{x^2+1}\,. \tag{77}$$

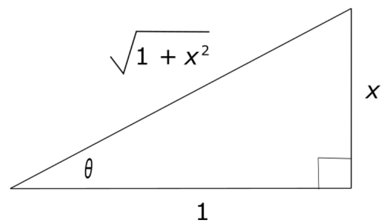

Figure 1. The values on the sides of the triangle are set so that $\tan\theta = x$.

According to the figure above,

$$x = \tan\theta \quad \text{and} \quad \theta = \tan^{-1} x\,, \tag{78}$$

To find $D_x \tan^{-1}$, I will compose functions and then differentiate, using the chain rule. We begin with

$$\tan(\tan^{-1} x) = x\,. \tag{79}$$

Differentiating by x gives

$$(D_\theta \tan\theta) D_x(\tan^{-1} x) = 1\,. \tag{80}$$

But $(D_\theta \tan\theta) = \sec^2\theta$, and,

$$\sec^2\theta = \tan^2\theta + 1 = x^2 + 1\,. \tag{81}$$

Therefore,

$$D_x \tan^{-1} x = \frac{1}{x^2+1}\,. \tag{82}$$

Problem 5:

Given that

$$D_x \tan^{-1} x = \frac{1}{x^2+1}\,. \tag{83}$$

find the integral

$$I = \int \tan^{-1} x\, dx\,. \tag{84}$$

We'll try the Carrington equation

$$D_x[\, x\tan^{-1} x\,] = \tan^{-1} x + \frac{x}{x^2+1}\,. \tag{85}$$

On integration, we get

$$x\tan^{-1} x = I + \int \frac{x}{x^2+1}\, dx\,. \tag{86}$$

But

$$\int \frac{x\,dx}{x^2+1} = \tfrac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \tfrac{1}{2}\ln\left|x^2+1\right|\,. \tag{87}$$

Therefore,

$$I = x\,\tan^{-1} x - \tfrac{1}{2}\ln\left|x^2+1\right| + C\,. \tag{88}$$

Problem 6:

Find the integral

$$I = \int \frac{1}{(a^2+x^2)^2}\,dx\,. \tag{89}$$

We begin with a Carrington equation:

$$D_x\Big[\,x\,\frac{1}{a^2+x^2}\,\Big] = \frac{1}{a^2+x^2} - \frac{2x^2}{(a^2+x^2)^2}\,. \tag{90}$$

On integrating, we have:

$$\frac{x}{a^2+x^2} = \int \frac{1}{a^2+x^2}\,dx - 2\int \frac{x^2}{(a^2+x^2)^2}\,dx\,. \tag{91}$$

By use of virtual emplacement on the last term:

$$\begin{aligned}\int \frac{x^2}{(a^2+x^2)^2}\,dx &= \int \frac{a^2+x^2}{(a^2+x^2)^2}\,dx - \int \frac{a^2}{(a^2+x^2)^2}\,dx \\ &= \int \frac{dx}{a^2+x^2} - a^2 I \\ &= \frac{1}{a}\tan^{-1}\frac{x}{a} - a^2 I\,. \end{aligned} \tag{92}$$

Substituting this last result into (91), we have that

$$\begin{aligned}\frac{x}{a^2+x^2} &= \frac{1}{a}\tan^{-1}\frac{x}{a} - 2\big[\frac{1}{a}\tan^{-1}\frac{x}{a} - a^2 I\big] \\ &= -\frac{1}{a}\tan^{-1}\frac{x}{a} + 2a^2 I\,. \end{aligned} \tag{93}$$

Solving for I gives:

$$I = \frac{1}{2a^2}\big[\frac{x}{a^2+x^2} + \frac{1}{a}\tan^{-1}\frac{x}{a}\,\big] + C\,. \tag{94}$$

Problem 7:

Find the integral

$$I = \int x^2 \cos x\,dx\,. \tag{95}$$

We begin with a Carrington equation:

$$D_x[\,x^2\sin x\,] = 2x\sin\,x + x^2\cos x\,, \tag{96}$$

and then integrate:

$$x^2\sin x = 2\int x\sin\,x\,dx + I\,. \tag{97}$$

Solving for I and using (36b), we have

$$I = x^2\sin x - 2[\sin x - x\cos x\,] + C\,. \tag{98}$$

Problem 8:

Find the integral

$$I = \int (\sin^{-1} x)^2 \, dx \, . \tag{99}$$

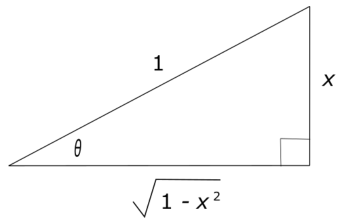

Figure 2. Standard set up of angles and sides.

Referring to the figure above, we set $\sin\theta = x$ then $dx = \cos\theta d\theta$:

$$\begin{aligned} I(\theta) &= \int \theta^2 \cos\theta \, d\theta \\ &= \theta^2 \sin\theta - 2(\sin\theta - \theta\cos\theta) + C \, . \end{aligned} \tag{100}$$

Therefore,

$$I(x) = (\sin^{-1} x)^2 x - 2x + 2(\sin^{-1} x)\sqrt{1 - x^2}\,] + C \, . \tag{101}$$

Problem 9:

Find the integral

$$I = \int \frac{1}{x - x^{3/5}} \, dx \, . \tag{102}$$

Okay, a variable substitution is in order here. Let

$$x = u^5 \quad \text{then} \quad dx = 5u^4 \, du \, . \tag{103}$$

Then

$$\begin{aligned} I(u) &= \int \frac{5u^4}{u^5 - u^3} \, du \\ &= 5 \int \frac{u}{u^2 - 1} \, du \\ &= \frac{5}{2} \int \frac{d(u^2 - 1)}{u^2 - 1} \\ &= \frac{5}{2} \ln(u^2 - 1) \, . \end{aligned} \tag{104}$$

Therefore,

$$I(x) = \frac{5}{2} \ln(x^{2/5} - 1) + C \, . \tag{105}$$

Problem 10:

Find the integral

$$I = \int \frac{\sin^{-1} x}{\sqrt{1+x}}\, dx\,. \tag{106}$$

Let's try a Carrington equation (using Equation (5)):

$$D_x[\sqrt{1+x}\,\sin^{-1} x\,] = \tfrac{1}{2}\frac{\sin^{-1} x}{\sqrt{1+x}} + \frac{\sqrt{1+x}}{\sqrt{1-x^2}}\,. \tag{107}$$

On simplifying and integrating, we get

$$\begin{aligned} \sqrt{1+x}\,\sin^{-1} x &= \tfrac{1}{2} I + \int \frac{dx}{\sqrt{1-x}} \\ &= \tfrac{1}{2} I - 2\sqrt{1-x}\,. \end{aligned} \tag{108}$$

Therefore,

$$I = 2\sqrt{1+x}\,\sin^{-1} x + 4\sqrt{1-x} + C\,. \tag{109}$$

Problem 11:

Find the integral

$$I = \int \csc\, x\, dx\,. \tag{110}$$

Hint: This problem is mostly trigonometric manipulation.

$$\begin{aligned} \int \csc\, x\, dx &= \int \frac{1}{\sin x}\, dx \\ &= \int \frac{\sin x}{\sin^2 x}\, dx \\ &= \int \frac{\sin x}{1-\cos^2 x}\, dx \\ &= \int \frac{\sin x}{(1-\cos x)(1+\cos x)}\,. \end{aligned} \tag{111}$$

By use of partial fractions, we get that

$$\frac{1}{(1-\cos x)(1+\cos x)} = \tfrac{1}{2}\Big[\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\Big]\,. \tag{112}$$

Hence,

$$\begin{aligned} I &= \tfrac{1}{2}\int \sin x\,\big[\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\,\big]\, dx \\ &= \tfrac{1}{2}\int \frac{\sin x}{1+\cos x}\, dx + \tfrac{1}{2}\int \frac{\sin x}{1-\cos x}\, dx \\ &= -\tfrac{1}{2}\int \frac{d(1+\cos x)}{1+\cos x} + \tfrac{1}{2}\int \frac{d(1-\cos x)}{1-\cos x} \\ &= -\tfrac{1}{2}\ln|1+\cos x| + \tfrac{1}{2}\ln|1-\cos x|\,. \end{aligned} \tag{113}$$

But,

$$-\tfrac{1}{2}\ln|1+\cos x| + \tfrac{1}{2}\ln|1-\cos x| = -\tfrac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right|$$
$$= -\tfrac{1}{2}\ln\left|\frac{(1+\cos x)^2}{1-\cos^2 x}\right|$$
$$= -\tfrac{1}{2}\ln\left|\frac{(1+\cos x)^2}{\sin^2 x}\right|$$
$$= -\ln\left|\frac{1+\cos x}{\sin x}\right|$$
$$= -\ln|\csc x + \cot x|\,. \tag{114}$$

Hence,

$$\int \csc\, x\, dx = -\ln|\csc x + \cot x|\,. \tag{115}$$

Problem 12:

Find the integral

$$I = \int \ln(x+\sqrt{x})\, dx\,. \tag{116}$$

First, a variable substitution.

$$x = u^2 \quad \text{hence} \quad dx = 2udu\,. \tag{117}$$

So,

$$I(u) = 2\int \ln(u^2+u)\, udu\,. \tag{118}$$

Now for the Carrington equation:

$$D_u[u^2\ln(u^2+u)] = 2u\ln(u^2+u) + \frac{u^2(2u+1)}{u^2+u}\,. \tag{119}$$

On simplifying and integrating, we have

$$u^2\ln(u^2+u) = I(u) + \int \frac{u(2u+1)}{u+1}\, du\,. \tag{120}$$

But (by long division)

$$\frac{u(2u+1)}{u+1} = 2u - 1 + \frac{1}{u+1}\,. \tag{121}$$

Hence,

$$I(u) = u^2\ln(u^2+u) - \int\left(2u - 1 + \frac{1}{u+1}\right) du$$
$$= u^2\ln(u^2+u) - u^2 + u - \ln|u+1| + C'\,. \tag{122}$$

Therefore,

$$\int \ln(x+\sqrt{x})\, dx = x\ln(x+\sqrt{x}) - x + \sqrt{x} - \ln|\sqrt{x}+1| + C\,. \tag{123}$$

Problem 13:

Find the integral

$$I = \int \tan^3 x \, dx \,. \tag{124}$$

We'll use mostly trig manipulations.

$$\begin{aligned} I &= \int (\tan x)(\tan^2 x) \, dx \\ &= \int (\tan x)(\sec^2 x - 1) \, dx \\ &= -\int \tan x \, dx + \int \tan x \sec^2 x \, dx \\ &= -\int \tan x \, dx + \int \sec x \, d(\sec x) \, dx \\ &= \ln|\cos x| + \tfrac{1}{2} \sec^2 x + C \,. \end{aligned} \tag{125}$$

Problem 14:

Find the integral

$$I = \int \frac{dx}{x\sqrt{x^2 + x + 1}} \,. \tag{126}$$

This time, we use a variable substitution.

$$x = \frac{1}{t} \quad \text{therefore} \quad dx = -\frac{dt}{t^2} \,. \tag{127}$$

On substituting into (126), we get

$$\begin{aligned} I(t) &= \int \frac{dt}{\sqrt{t^2 + t + 1}} \\ I &= \int \frac{dt}{\sqrt{(t + 1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} \quad (\text{"complete the square"}) \\ &= \ln|\, t + 1/2 + \sqrt{(t + 1/2)^2 + 3/4}\,| + C \quad [\text{using (35)}] \,. \end{aligned} \tag{128}$$

Therefore,

$$\int \frac{dx}{x\sqrt{x^2 + x + 1}} = \ln\left|\, \frac{1 + x/2 + \sqrt{x^2 + x + 1}}{x} \,\right| + C \,. \tag{129}$$

Problem 15:

Find the integral

$$I = \int \sinh^3 x \cosh^2 x \, dx \,. \tag{130}$$

Let's use the Carrington

$$D_x[\sinh^2 x \; \cosh^3 x\,] = 2 \sinh x \cosh^4 x + 3 \sinh^3 x \cosh^2 x \,. \tag{131}$$

Integrating, we get

$$\sinh^2 x \; \cosh^3 x = 2 \int \sinh x \; \cosh^4 x \, dx + 3I \,. \tag{132}$$

But

$$\int \sinh x \cosh^4 x \, dx = \int \cosh^4 x \, d(\cosh x) = \tfrac{1}{5} \cosh^5 x + C' \,. \tag{133}$$

Plugging this into (132), we have that

$$\sinh^2 x \, \cosh^3 x = \tfrac{2}{5} \cosh^5 x + 3I + C'' \,. \tag{134}$$

Solving for I, we get

$$I = \tfrac{1}{3}[\sinh^2 x \, \cosh^3 x - \tfrac{2}{5} \cosh^5 x] + C \,. \tag{135}$$

Problem 16:

Find the integral

$$I = \int \frac{1}{x^2 + \sqrt{2}x + 1} \, dx \,. \tag{136}$$

First, complete the square in the denominator:

$$I = \int \frac{1}{(x + \frac{1}{\sqrt{2}})^2 + \frac{1}{2}} \, dx = \int \frac{d(x + \frac{1}{\sqrt{2}})}{(x + \frac{1}{\sqrt{2}})^2 + \frac{1}{2}} \,. \tag{137}$$

So, using table integral (9), we get

$$\int \frac{dx}{x^2 + \sqrt{2}x + 1} = \sqrt{2} \tan^{-1}[\sqrt{2}(x + \frac{1}{\sqrt{2}})] + C \,. \tag{138}$$

6 Appendix

For those who want to get really good at integration over the real numbers, you are going to have master the delicacies of logs and hyperbolic trig functions. The following problem is designed to give you some practice at going between them.

Convert $\tanh^{-1} x$ into a natural logarithm of some function of x.

Let

$$u = \tanh^{-1} x \quad \text{for} \quad -1 < x < 1 \,. \tag{139}$$

Then

$$x = \tanh \, u \,. \tag{140}$$

But

$$\tanh \, u = \frac{e^u + e^{-u}}{e^u - e^{-u}} = \frac{e^{2u} + 1}{e^{2u} - 1} \,. \tag{141}$$

For algebraic simplicity, let's set $y = e^u$. Then Equation (140) becomes

$$x = \frac{y^2 + 1}{y^2 - 1} \,. \tag{142}$$

On solving for y, we get

$$y = \sqrt{\frac{x + 1}{x - 1}} \,. \tag{143}$$

On taking the natural logs of both sides, we get

$$u = \tfrac{1}{2} \ln \left| \frac{x + 1}{x - 1} \right| \,. \tag{144}$$

Hence,

$$\tanh^{-1} x = \tfrac{1}{2} \ln \left| \frac{x + 1}{x - 1} \right| \quad \text{for} \quad -1 < x < 1 \,. \tag{145}$$