

Integration Techniques Paper 4

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June 21, 2025

Abstract

Herein is a collection of integration techniques for indefinite integrals. I found the problems from a variety of sources, most of which I no longer know where they came from.

1 Introduction

Indefinite integrals, when they exist, result in functions, whereas, definite integrals, when they exist, result in numbers. The purpose of this paper is to present some techniques to perform indefinite integrals. My major strategy will be to reduce or convert a given integral into a form which is in a robust integral table, though, I may attempt to reduce the integral as far as I can take it. The integrals are presented in no particular order.

Warning: This paper assumes that the reader has a basic knowledge of integration (such as integration by parts), along with a familiarity with trigonometric and hyperbolic identities, logarithms, exponentials, partial fractions – generally speaking, the stuff found in a course on Algebra 2.

Note: The symbol D_x means to differentiate by x .

2 Table of Integrals and Derivatives for Later Use

First, the derivatives:

$$D_x \sin x = \cos x . \tag{1}$$

$$D_x \cos x = -\sin x . \tag{2}$$

$$D_x \sec x = \sec x \tan x . \tag{3}$$

$$D_x \tan x = \sec^2 x . \tag{4}$$

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} , \tag{5}$$

Some hyperbolic trigonometric functions and derivatives:

$$\cosh^2 x - \sinh^2 x = 1 . \tag{6}$$

$$\sinh^2 x = \frac{1}{2}[-1 + \cosh(2x)] . \tag{7}$$

$$\cosh^2 x = \frac{1}{2}[1 + \cosh(2x)] . \tag{8}$$

$$\sinh(2x) = 2 \sinh x \cosh x . \tag{9}$$

$$\cosh(2x) = \sinh^2 x + \cosh^2 x . \tag{10}$$

$$D_x \cosh x = \sinh x . \quad (11)$$

$$D_x \sinh x = \cosh x . \quad (12)$$

Now, the integrals:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C . \quad (13)$$

$$\int \frac{1}{(a^2 + x^2)^2} dx = \frac{1}{2a^2} \left[\frac{x}{a^2 + x^2} + \frac{1}{a} \tan^{-1} \frac{x}{a} \right] + C . \quad (14)$$

$$\int \frac{du}{u} = \ln u + C . \quad (15)$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C . \quad (16)$$

$$\int \csc x dx = -\ln |\csc x + \cot x| . \quad (17)$$

$$\int \csc^2 x dx = -\cot x + C . \quad (18)$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C . \quad (19)$$

$$\int \ln x dx = x \ln x - x + C . \quad (20)$$

$$\int x e^x dx = e^x (x - 1) + C . \quad (21)$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C . \quad (22)$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C . \quad (23)$$

$$\int \frac{1}{x \sqrt{a^2 - x^2}} dx = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x} + C . \quad (24)$$

$$\int \frac{x^2}{1 + x^2} dx = x - \tan^{-1} x + C , \quad (25)$$

$$\int \tan x dx = -\ln |\cos x| + C = \ln |\sec x| + C . \quad (26)$$

$$\int \tan^2 x dx = \tan x - x + C . \quad (27)$$

$$\int \tan^3 x dx = \ln |\cos x| + \frac{1}{2} \sec^2 x + C . \quad (28)$$

$$\int \cot x dx = \ln |\sin x| + C . \quad (29)$$

$$\int \cot^2 x \, dx = -x - \cot x + C. \quad (30)$$

$$\int \ln(a^2 + x^2) \, dx = x \ln(a^2 + x^2) + 2x - 2a \tan^{-1} \left(\frac{x}{a} \right) + C. \quad (31)$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln |\sqrt{1+x^2} + x| + C. \quad (32)$$

$$\int \sqrt{1+x^2} \, dx = \frac{1}{2} [\sinh^{-1} x + x\sqrt{1+x^2}] + C. \quad (33)$$

$$\int \sqrt{1-x^2} \, dx = \frac{1}{2} [\sin^{-1} x + x\sqrt{1-x^2}] + C. \quad (34)$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C. \quad (35)$$

$$\int \frac{x}{\sqrt{x^2 - a^2}} \, dx = \sqrt{x^2 - a^2} + C. \quad (36)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C. \quad (37)$$

$$\int \frac{\sqrt{x^2 - a^2}}{x} \, dx = \sqrt{x^2 - a^2} - \cos^{-1} \frac{a}{x} + C. \quad (38)$$

$$\int x^2 \ln x \, dx = \frac{1}{3} x^3 [\ln x - \frac{1}{3}] + C. \quad (39)$$

$$\int x e^x \, dx = e^x (x - 1) + C. \quad (40)$$

$$\int x^2 e^x \, dx = e^x (x^2 - 2x + 2) + C. \quad (41)$$

$$\int x^3 e^x \, dx = e^x (x^3 - 3x^2 + 6x - 6) + C. \quad (42)$$

$$\int x^3 e^{-x^2} \, dx = -\frac{1}{2} e^{-x^2} (x^2 + 1) + C. \quad (43)$$

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2} \frac{x}{1+x^2} + \frac{1}{2} \tan^{-1} x + C. \quad (44)$$

$$\int \frac{\sqrt{a^2 - x^2}}{x} \, dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + C. \quad (45)$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left| x + \sqrt{x^2 + a^2} \right| + C. \quad (46)$$

$$\int x \cos x \, dx = \cos x + x \sin x + C_1, \quad (47a)$$

$$\int x \sin x \, dx = \sin x - x \cos x + C_2. \quad (47b)$$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2(\cos x + x \sin x) + C, \quad (48a)$$

$$\int x^2 \cos x \, dx = x^2 \sin x - 2[\sin x - x \cos x] + C. \quad (48b)$$

$$\int e^x \sin x \, dx = \frac{1}{2}e^x(\sin x - \cos x), \quad (49a)$$

$$\int e^x \cos x \, dx = \frac{1}{2}e^x(\sin x + \cos x). \quad (49b)$$

$$\int \frac{1}{1 + \cos x} \, dx = -\cot x + \csc x + C. \quad (50)$$

$$\int \sin(ax) \cos(bx) \, dx = -\frac{b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)}{a^2 - b^2} + C_1 \quad (51a)$$

$$\int \cos(ax) \sin(bx) \, dx = \frac{a \sin(ax) \sin(bx) + b \cos(ax) \cos(bx)}{a^2 - b^2} + C_2. \quad (51b)$$

$$\int \frac{dz}{\sqrt{z^2 - 1}} = \cosh^{-1} z + C = \ln(z + \sqrt{z^2 - 1}) + C \quad (z > 1). \quad (52)$$

$$\int \ln(z + \sqrt{z^2 - 1}) \, dz = z \ln(z + \sqrt{z^2 - 1}) - \sqrt{z^2 - 1} + C. \quad (53)$$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C. \quad (54)$$

$$\int (\sin^{-1} x)^2 \, dx = (\sin^{-1} x)^2 x - 2x + 2(\sin^{-1} x)\sqrt{1 - x^2} + C. \quad (55)$$

$$\int \frac{dx}{x\sqrt{x^2 + x + 1}} = \ln \left| \frac{1 + x/2 + \sqrt{x^2 + x + 1}}{x} \right| + C. \quad (56)$$

$$\int \frac{1}{x^2 + \sqrt{2}x + 1} \, dx = \sqrt{2} \tan^{-1}[\sqrt{2}x + 1] + C, \quad (57a)$$

$$\int \frac{1}{x^2 - \sqrt{2}x + 1} \, dx = \sqrt{2} \tan^{-1}[\sqrt{2}x - 1] + C. \quad (57b)$$

$$\int \frac{x}{x^2 + \sqrt{2}x + 1} \, dx = \frac{1}{2} \ln(x^2 + \sqrt{2}x + 1) - 2 \tan^{-1}[\sqrt{2}x + 1] + C, \quad (58a)$$

$$\int \frac{x}{x^2 - \sqrt{2}x + 1} \, dx = \frac{1}{2} \ln(x^2 - \sqrt{2}x + 1) + 2 \tan^{-1}[\sqrt{2}x - 1] + C. \quad (58b)$$

3 Virtual Emplacement

So much of mathematics employs tricks that get used over and over in a wide variety of subject areas, yet go unnamed, and thus are hard to explain to one's readers when one uses them. Decades ago I invented the term *virtual emplacement* to refer to the algebraic action of adding a zero to an expression, or multiplying or dividing an expression by unity, or more generally, performing some function and its inverse to an expression, such as

$$x\sqrt{x+y} = \sqrt{x^2(x+y)} \quad \text{when } x, y \geq 0. \quad (59)$$

Let's look at the expression $\frac{x^2}{1+x^2}$. Can we simplify it?¹ Yes, by performing a virtual emplacement.

$$\frac{x^2}{1+x^2} = \frac{x^2+1-1}{1+x^2} = \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}. \quad (60)$$

Stuff like this comes up all the time in integration.

Now, as a real application, consider the integral:

$$\int \frac{x^2}{1+x^2} dx = ? \quad (61)$$

Clearly, we can use the table integral (13), if we can message it into the correct form. But we can do that by employing the result of (60), as well, to get

$$\begin{aligned} \int \frac{x^2}{1+x^2} dx &= \int 1 dx - \int \frac{dx}{1+x^2}, \\ \text{hence } \int \frac{x^2}{1+x^2} dx &= x - \tan^{-1} x + C, \end{aligned} \quad (62)$$

where C is an arbitrary constant.

4 The 'Carrington' (Differential) Equation vs. Integration by Parts

Integration by Parts is one of the most used techniques in the bag of gimmicks of indefinite integration. The technique provides us with an integration identity:

$$\int u dv = vu - \int v du. \quad (63)$$

The proof of it is based on the product rule of differentiation:

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x). \quad (64)$$

Now, we integrate:

$$f(x)g(x) = \int g(x)D_x f(x) dx + \int f(x)D_x g(x) dx, \quad (65)$$

or,

$$f(x)g(x) = \int g(x) df + \int f(x) dg, \quad (66)$$

¹By 'simplify' in this case, I mean to reduce the given expression to a sum of expressions, each of which is easier to integrate than the original expression.

Now, let $u = f(x)$ and $v = g(x)$, then this last equation becomes

$$uv = \int v du + \int u dv, \quad (67)$$

from which follows (63).

The Carrington² differential equation, associated to some integral, is any equation of the form³

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x) \quad (68)$$

that facilitates an integration problem. The next step is to integrate across (68).

This Carrington equation is not unique, though I will sometimes proffer to the reader ‘the Carrington equation’, which should be interpreted merely as the *particular* Carrington equation that I chose.

Heuristic: I often set $g(x)$ to be the integrand of the integral and then set $f(x) = x$.

Let’s now do an example problem: Find the integral⁴

$$I = \int x \ln x dx, \quad (69)$$

We start with the Carrington differential equation (which follows the above heuristic):

$$D_x[x^2 \ln x] = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x. \quad (70)$$

Now we integrate:

$$\begin{aligned} x^2 \ln x &= 2I + \int x dx \\ &= 2I + \frac{1}{2}x^2. \end{aligned} \quad (71)$$

From this we get that

$$2I = x^2 \ln x - \frac{1}{2}x^2 + C', \quad (72)$$

or

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C. \quad (73)$$

²For the moment, this name is a place holder until I can come up with a better name for it. Dr. Carrington refers to a fictional character in the scifi movie *The Thing from Another World*, 1951. This equation is merely an ansatz and is, in most cases, not unique.

³The LHS of the Carrington Equation could contain a product of three or more factors, but that gets messy fast.

⁴I will often use the variables I, J, K , etc as placeholders for integrals to reduce the visual mess of the problem.

5 The Integrals

Problem 1:

Using partial fractions, find the integral

$$I = \int \frac{1}{x^3 + x^2 + x + 1} dx. \quad (74)$$

First, it's easy to show that

$$x^3 + x^2 + x + 1 = (x + 1)(x^2 + 1). \quad (75)$$

Then,

$$\frac{1}{(x + 1)(x^2 + 1)} = \frac{1 - x}{2(x^2 + 1)} + \frac{1}{2(x + 1)} \quad (76)$$

On integrating, we get

$$\begin{aligned} I &= -\frac{1}{2} \int \frac{x - 1}{x^2 + 1} dx + \frac{1}{2} \int \frac{1}{x + 1} dx \\ &= -\frac{1}{2} \int \frac{x}{x^2 + 1} dx + \frac{1}{2} \int \frac{1}{x^2 + 1} dx + \frac{1}{2} \int \frac{1}{x + 1} dx \\ &= -\frac{1}{4} \int \frac{d(x^2 + 1)}{x^2 + 1} dx + \frac{1}{2} \int \frac{1}{x^2 + 1} dx + \frac{1}{2} \int \frac{1}{x + 1} dx \\ &= -\frac{1}{4} \ln(x^2 + 1) + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \ln(x + 1) + C \end{aligned} \quad (77)$$

Problem 2:

Using partial fractions, find the integral

$$I = \int \sin(\ln x) dx. \quad (78)$$

Let's foresee the need to define the following, as well

$$J = \int \cos(\ln x) dx. \quad (79)$$

Now, we demonstrate two Carringtons. First,

$$D_x[x \sin(\ln x)] = \sin(\ln x) + \cos(\ln x). \quad (80)$$

On integrating, we get

$$x \sin(\ln x) = I + J. \quad (81)$$

Next,

$$D_x[x \cos(\ln x)] = \cos(\ln x) - \sin(\ln x). \quad (82)$$

On integrating, we get

$$x \cos(\ln x) = J - I. \quad (83)$$

Solving (81) and (83) together, we have

$$I = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C, \quad (84a)$$

$$J = \frac{1}{2}x[\sin(\ln x) + \cos(\ln x)] + C. \quad (84b)$$

Problem 3:

Find the integral

$$I = \int \frac{x^7}{x^4 + 1} dx. \quad (85)$$

First, a virtual emplacement,

$$\begin{aligned} \int \frac{x^7}{x^4 + 1} dx &= \int x^3 \frac{(x^4 + 1) - 1}{x^4 + 1} dx \\ &= \int x^3 dx - \int \frac{x^3}{x^4 + 1} dx \\ &= \int x^3 dx - \frac{1}{4} \int \frac{d(x^4 + 1)}{x^4 + 1} \\ &= \frac{1}{4} x^4 - \frac{1}{4} \ln(x^4 + 1) + C. \end{aligned} \quad (86)$$

Problem 4:

Find the integral

$$I = \int x\sqrt{x+1} dx. \quad (87)$$

First, a virtual emplacement,

$$\begin{aligned} \int x\sqrt{x+1} dx &= \int (x+1)\sqrt{x+1} dx - \int \sqrt{x+1} dx \\ &= \int (x+1)^{3/2} dx - \int (x+1)^{1/2} dx \\ &= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C. \end{aligned} \quad (88)$$

Lemma 1:

Show that

$$\cosh^{-1} x = \ln |x + \sqrt{x^2 - 1}|. \quad (89)$$

Proof:

Let

$$u = \cosh^{-1} x. \quad (90)$$

Then

$$x = \cosh u = \frac{e^u + e^{-u}}{2} = \frac{e^{2u} + 1}{2e^u}. \quad (91)$$

For algebraic simplicity, let $y = e^u$, then

$$x = \frac{y^2 + 1}{2y}. \quad (92)$$

From which we get the quadratic equation

$$y^2 - 2xy + 1 = 0, \quad (93)$$

which has solution

$$y = x + \sqrt{x^2 - 1}. \quad (94)$$

Therefore, since $u = \ln y$,

$$u = \ln |x + \sqrt{x^2 - 1}|. \quad (95)$$

Thus we have established (89).

Problem 5:

Find the integral

$$I = \int \sqrt{x^2 - 1} \, dx. \quad (96)$$

(We shall present two versions of the answer.)

First, a variable substitution:

$$\begin{aligned} x &= \cosh u, \\ \sqrt{x^2 - 1} &= \sinh u, \\ dx &= \sinh u \, du. \end{aligned} \quad (97)$$

Therefore,

$$I(u) = \int \sinh^2 u \, du. \quad (98)$$

This is easier to integrate if we employ identity (7):

$$\begin{aligned} I(u) &= \int \frac{1}{2}[-1 + \cosh(2u)] \, du \\ &= -\frac{1}{2}u + \frac{1}{4} \int \cosh(2u) \, d(2u) \\ &= -\frac{1}{2}u + \frac{1}{4} \sinh(2u) + C'' \\ &= -\frac{1}{2}u + \frac{1}{2} \sinh u \cosh u + C'' \quad (\text{using (9)}) \\ &= -\frac{1}{2} \cosh^{-1} x + \frac{1}{2}x\sqrt{x^2 - 1} + C'. \end{aligned} \quad (99)$$

So, on using the previous lemma, we get that

$$\int \sqrt{x^2 - 1} \, dx = -\frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + \frac{1}{2}x\sqrt{x^2 - 1} + C. \quad (100)$$

Problem 6:

Find the integral

$$I = \int x^3 \sqrt{1 - x^2} \, dx. \quad (101)$$

First, a simple identity:

$$x^3 = (-x)[(1 - x^2) - 1]. \quad (102)$$

Why did I do all that? To get the embedded expression $(1 - x^2)$. Thus

$$\begin{aligned} I &= \int (-x)(1 - x^2)^{3/2} \, dx + \int x\sqrt{1 - x^2} \, dx \\ &= \frac{1}{2} \int (1 - x^2)^{3/2} \, d(1 - x^2) - \frac{1}{2} \int (1 - x^2)^{1/2} \, d(1 - x^2) \\ &= \frac{1}{5}(1 - x^2)^{5/2} - \frac{1}{3}(1 - x^2)^{3/2} + C. \end{aligned} \quad (103)$$

Problem 7:

Given the so-called ‘Logarithmic integral’⁵

$$\operatorname{li}(x) \equiv \int \frac{dx}{\ln x}, \quad (104)$$

find the integral

$$I = \int \ln(\ln x) dx. \quad (105)$$

Let’s try the Carrington:

$$D_x[x \ln(\ln x)] = \ln(\ln x) + \frac{1}{\ln x}. \quad (106)$$

On integrating, we get

$$x \ln(\ln x) = I + \operatorname{li}(x) - C. \quad (107)$$

Hence,

$$\int \ln(\ln x) dx = x \ln(\ln x) - \operatorname{li}(x) + C. \quad (108)$$

Problem 8:

Find the integral

$$I = \int \sqrt{1 + e^{2x}} dx. \quad (109)$$

Let’s try a variable substitution:

$$\begin{aligned} v^2 &= 1 + e^{2x}, \\ 2v dv &= 2e^{2x} dx, \\ dx &= \frac{v dv}{v^2 - 1}. \end{aligned} \quad (110)$$

Then,

$$\begin{aligned} I(v) &= \int \frac{v^2 dv}{v^2 - 1} \\ &= \int \frac{(v^2 - 1) + 1}{v^2 - 1} dv \\ &= \int dv + \int \frac{1}{v^2 - 1} dv \\ &= v + \int \frac{1}{2} \ln \left| \frac{v - 1}{v + 1} \right| + C'. \end{aligned} \quad (111)$$

And, finally,

$$\int \sqrt{1 + e^{2x}} dx = \sqrt{1 + e^{2x}} + \frac{1}{2} \ln \left| \frac{\sqrt{1 + e^{2x}} - 1}{\sqrt{1 + e^{2x}} + 1} \right| + C. \quad (112)$$

⁵There is a distinction between $\operatorname{li}(x)$ and $\operatorname{Li}(x)$.

Problem 9:

Given that

$$\cot \frac{u}{2} = \csc u + \cot u, \quad (113)$$

find the integral

$$I = \int \frac{1}{\sec x - 1} dx. \quad (114)$$

First, multiply through by $\cos x$:

$$I = \int \frac{\cos x}{1 - \cos x} dx. \quad (115)$$

Now, multiply through (the fraction) by $1 + \cos x$ and use that $1 - \cos^2 x = \sin^2 x$:

$$\begin{aligned} I &= \int \frac{(\cos x)(1 + \cos x)}{\sin^2 x} dx \\ &= \int \frac{(\cos x)}{\sin^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x} dx \\ &= \int \frac{d(\sin x)}{\sin^2 x} dx + \int \cot^2 x dx \\ &= -\frac{1}{\sin x} - x - \cot x + C \\ &= -x - \cot \frac{x}{2} + C. \end{aligned} \quad (116)$$

Problem 10:

Find the integral

$$I = \int \frac{2x + 1}{\sqrt{x + 4}} dx. \quad (117)$$

Let's use virtual emplacement:

$$\begin{aligned} \int \frac{2x + 1}{\sqrt{x + 4}} dx &= \int \frac{2(x + 4) - 7}{\sqrt{x + 4}} dx \\ &= 2 \int \frac{(x + 4)}{\sqrt{x + 4}} dx - 7 \int \frac{1}{\sqrt{x + 4}} dx \\ &= 2 \int (x + 4)^{1/2} dx - 7 \int (x + 4)^{-1/2} dx \\ &= \frac{4}{3}(x + 4)^{3/2} - 14(x + 4)^{1/2} + C \\ &= (x + 4)^{1/2} \left[\frac{4}{3}(x + 4) - 14 \right] + C \\ &= \frac{2}{3}(x + 4)^{1/2} [2(x + 4) - \frac{3}{2}14] + C \\ &= \frac{2}{3}(x + 4)^{1/2} [2x - 13] + C. \end{aligned} \quad (118)$$

Problem 11:

Find the integral

$$I = \int (x + 1)e^{x^2 + 2x} dx. \quad (119)$$

The fastest way to integrate this, if we can do it, would be to match it against

$$I = \int u e^{u^2} du = \frac{1}{2} e^{u^2} + C. \quad (120)$$

Let's use virtual emplacement:

$$\begin{aligned} \int (x+1)e^{x^2+2x} dx &= \int (x+1)e^{(x^2+2x+1)-1} dx \\ &= e^{-1} \int (x+1)e^{(x+1)^2} dx \\ &= e^{-1} \int (x+1)e^{(x+1)^2} d(x+1) \\ &= \frac{1}{2e} e^{(x+1)^2} + C \\ &= \frac{1}{2} e^{x^2+2x} + C. \end{aligned} \quad (121)$$

Problem 12:

Find the integral

$$I = \int \sqrt{1 - \cos x} dx. \quad (122)$$

At this point, the steps should be easy to follow.

$$\begin{aligned} \int \sqrt{1 - \cos x} dx &= \int \frac{\sqrt{1 - \cos^2 x}}{\sqrt{1 + \cos x}} dx \\ &= \int \frac{\sin x}{\sqrt{1 + \cos x}} dx \\ &= - \int \frac{d(1 + \cos x)}{\sqrt{1 + \cos x}} \\ &= -2\sqrt{1 + \cos x} + C. \end{aligned} \quad (123)$$

Problem 13:

Find the integral

$$I = \int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} dx. \quad (124)$$

Let's try a Carrington equation:

$$D_x \left[\frac{x^2 e^{x^2}}{(x^2 + 1)} \right] = \frac{(2x + 2x^3)e^{x^2}}{(x^2 + 1)} + \frac{(x^2)(-2x)e^{x^2}}{(x^2 + 1)^2}. \quad (125)$$

On integrating, we get

$$\frac{x^2 e^{x^2}}{(x^2 + 1)} = \int 2x e^{x^2} dx + -2I. \quad (126)$$

Hence

$$\int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} dx = \frac{e^{x^2}}{2} \left[1 - \frac{x^2}{(x^2 + 1)} \right] + C = \frac{e^{x^2}}{2(x^2 + 1)} + C. \quad (127)$$

6 Appendix:

The fast way to factor $x^4 + 1$ into a product of quadratics (using VE):

$$\begin{aligned}x^4 + 1 &= (x^4 + 2x^2 + 1) - 2x^2 \\&= (x^2 + 1)^2 - (\sqrt{2}x)^2 \\&= [(x^2 + 1) + \sqrt{2}x][(x^2 + 1) - \sqrt{2}x]. \\&= [x^2 + \sqrt{2}x + 1][x^2 - \sqrt{2}x + 1].\end{aligned}\tag{128}$$