Integration Techniques Paper 5

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Abstract

Herein is a collection of integration techniques for indefinite integrals. I found the problems from a variety of sources, most of which I no longer know where they came from. Herein, is introduced the Weierstrass Substitution for integration.

1 Introduction

Indefinite integrals, when they exist, result in functions, whereas, definite integrals, when they exist, result in numbers. The purpose of this paper is to present some techniques to perform indefinite integrals. My major strategy will be to reduce or convert a given integral into a form which is in a robust integral table, though, I may attempt to reduce the integral as far as I can take it. The integrals are presented in no particular order.

Warning: This paper assumes that the reader has a basic knowledge of integration (such as integration by parts), along with a familiarity with trigonometric and hyperbolic identities, logarithms, exponentials, partial fractions – generally speaking, the stuff found in a course on Algebra 2.

Note: The symbol D_x means to differentiate by x.

2 Table of Integrals and Derivatives for Later Use

First, the derivatives:

$$D_x \sin x = \cos x\,. \tag{1}$$

$$D_x \cos x = -\sin x\,. \tag{2}$$

$$D_x \sec x = \sec x \tan x\,. \tag{3}$$

$$D_x \tan x = \sec^2 x\,. \tag{4}$$

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}\,, \tag{5}$$

$$D_x \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}\,, \tag{6}$$

$$D_x \tan^{-1} x = \frac{1}{x^2+1}\,, \tag{7}$$

$$D_x \cot^{-1} x = -\frac{1}{x^2+1}\,, \tag{8}$$

$$D_x \sec^{-1} x = \frac{1}{|\,x\,|\sqrt{x^2-1}}\,, \tag{9}$$

$$D_x \csc^{-1} x = -\frac{1}{|\,x\,|\sqrt{x^2-1}}\,, \tag{10}$$

Some hyperbolic and trigonometric functions and derivatives:

$$\sin\frac{x}{2} = \pm\frac{\sqrt{1-\cos x}}{2}\,. \tag{11}$$

$$\cos\frac{x}{2} = \pm\frac{\sqrt{1+\cos x}}{2}\,. \tag{12}$$

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}\,. \tag{13}$$

$$\sin\,2x = 2\sin x\,\cos x\,. \tag{14}$$

$$\cos\,2x = \cos^2 x - \sin^2 x\,. \tag{15}$$

$$\tan\,2x = \frac{2\tan\,x}{1-\tan^2 x}\,. \tag{16}$$

$$\sin^2 x = \tfrac{1}{2}(1+\cos 2x)\,. \tag{17}$$

$$\cos^2 x = \tfrac{1}{2}(1-\cos 2x)\,. \tag{18}$$

$$\cosh^2 x - \sinh^2 x = 1\,. \tag{19}$$

$$\sinh^2 x = \tfrac{1}{2}[-1+\cosh(2x)]\,. \tag{20}$$

$$\cosh^2 x = \tfrac{1}{2}[1+\cosh(2x)]\,. \tag{21}$$

$$\sinh(2x) = 2\sinh x\cosh x\,. \tag{22}$$

$$\cosh\,(2x) = \sinh^2 x + \cosh^2 x\,. \tag{23}$$

$$\tanh\,\left(\frac{x}{2}\right) = \frac{\cosh x - 1}{\sinh x}\,. \tag{24}$$

$$D_x \cosh x = \sinh x\,. \tag{25}$$

$$D_x \sinh x = \cosh x\,. \tag{26}$$

Now, the integrals:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\,. \tag{27}$$

$$\int \frac{1}{(a^2+x^2)^2}\,dx = \frac{1}{2a^2}\left[\frac{x}{a^2+x^2} + \frac{1}{a}\tan^{-1}\frac{x}{a}\right] + C\,. \tag{28}$$

$$\int \frac{du}{u} = \ln u + C\,. \tag{29}$$

$$\int \sec x \, dx = \ln \big| \sec x + \tan x \big| + C \,. \tag{30}$$

$$\int \csc \, x \, dx = -\ln | \, \csc x + \cot x \, | \,. \tag{31}$$

$$\int \csc^2 x \, dx = -\cot x + C \,. \tag{32}$$

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C \,. \tag{33}$$

$$\int \ln x \, dx = x \ln x - x + C \,. \tag{34}$$

$$\int \ln(\ln x) \, dx = x \ln(\ln x) - \mathrm{li}(x) + C \,. \tag{35}$$

$$\int x e^x \, dx = e^x (x-1) + C \,. \tag{36}$$

$$\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \Big| \frac{x-a}{x+a} \Big| + C \,. \tag{37}$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C \,. \tag{38}$$

$$\int \frac{1}{x\sqrt{a^2 - x^2}} \, dx = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x} + C \,. \tag{39}$$

$$\int \frac{x^2}{1+x^2} \, dx = x - \tan^{-1} x + C \,, \tag{40}$$

$$\int \tan x \, dx = -\ln \big| \cos x \big| + C = \ln \big| \sec x \big| + C \,. \tag{41}$$

$$\int \tan^2 x \, dx = \tan x - x + C \,. \tag{42}$$

$$\int \tan^3 x \, dx = \ln | \cos x | + \tfrac{1}{2} \sec^2 x + C \,. \tag{43}$$

$$\int \cot x \, dx = \ln \big| \sin x \big| + C \,. \tag{44}$$

$$\int \cot^2 x \, dx = -x - \cot x + C \,. \tag{45}$$

$$\int \ln(a^2 + x^2) \, dx = x \ln(a^2 + x^2) + 2x - 2a \tan^{-1} \left(\frac{x}{a}\right) + C \,. \tag{46}$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln \big| \sqrt{1+x^2} + x \big| + C \,. \tag{47}$$

$$\int \sqrt{1+x^2} \, dx = \tfrac{1}{2} \big[\sinh^{-1} x + x\sqrt{1+x^2} \big] + C \,. \tag{48}$$

$$\int \sqrt{1-x^2}\,dx = \tfrac{1}{2}\big[\,\sin^{-1}x + x\sqrt{1-x^2}\,\big] + C\,. \tag{49}$$

$$\int \sqrt{x^2-1}\,dx = -\tfrac{1}{2}\ln|x+\sqrt{x^2-1}\,| + \tfrac{1}{2}x\sqrt{x^2-1} + C\,. \tag{50}$$

$$\int x^3\sqrt{1-x^2}\,dx = \tfrac{1}{5}(1-x^2)^{5/2} - \tfrac{1}{3}(1-x^2)^{3/2} + C\,. \tag{51}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C\,. \tag{52}$$

$$\int \frac{x}{\sqrt{x^2-a^2}}\,dx = \sqrt{x^2-a^2} + C\,. \tag{53}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C\,. \tag{54}$$

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - \cos^{-1}\frac{a}{x} + C\,. \tag{55}$$

$$\int x^2\ln x\,dx = \tfrac{1}{3}x^3[\,\ln x - \tfrac{1}{3}\,] + C\,. \tag{56}$$

$$\int xe^x\,dx = e^x(x-1) + C\,. \tag{57}$$

$$\int x^2e^x\,dx = e^x(x^2-2x+2) + C\,. \tag{58}$$

$$\int x^3e^x\,dx = e^x(x^3-3x^2+6x-6) + C\,. \tag{59}$$

$$\int x^3e^{-x^2}\,dx = -\tfrac{1}{2}e^{-x^2}(x^2+1) + C\,. \tag{60}$$

$$\int \frac{dx}{(1+x^2)^2}\,dx = \tfrac{1}{2}\frac{x}{1+x^2} + \tfrac{1}{2}\tan^{-1}x + C\,. \tag{61}$$

$$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C\,. \tag{62}$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x+\sqrt{x^2+a^2}\right| + C\,. \tag{63}$$

$$\int x\sqrt{x+1}\,dx = \tfrac{2}{5}(x+1)^{5/2} - \tfrac{2}{3}(x+1)^{3/2} + C\,. \tag{64}$$

$$\int x\cos x\,dx = \cos x + x\sin x + C_1\,, \tag{65a}$$

$$\int x\sin x\,dx = \sin x - x\cos x + C_2\,. \tag{65b}$$

$$\int x^2 \sin\, x\, dx = -x^2 \cos\, x + 2(\cos x + x \sin x) + C\,, \tag{66a}$$
$$\int x^2 \cos\, x\, dx = x^2 \sin x - 2[\,\sin x - x \cos x\,] + C\,. \tag{66b}$$

$$\int e^x \sin x\, dx = \tfrac{1}{2} e^x (\sin x - \cos x)\,, \tag{67a}$$
$$\int e^x \cos x\, dx = \tfrac{1}{2} e^x (\sin x + \cos x)\,. \tag{67b}$$

$$\int \frac{1}{1+\cos x}\, dx = -\cot x + \csc x + C\,. \tag{68}$$

$$\int \sin(ax)\cos(bx)\, dx \;= -\frac{b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)}{a^2 - b^2} + C_1 \tag{69a}$$
$$\int \cos(ax)\sin(bx)\, dx = \frac{a\sin(ax)\sin(bx) + b\cos(ax)\cos(bx)}{a^2 - b^2} + C_2\,. \tag{69b}$$

$$\int \frac{dz}{\sqrt{z^2-1}} = \cosh^{-1} z + C = \ln\left(z + \sqrt{z^2-1}\right) + C \quad (z > 1)\,. \tag{70}$$

$$\int \ln\left(z + \sqrt{z^2-1}\right) dz = z \ln\left(z + \sqrt{z^2-1}\right) - \sqrt{z^2-1} + C\,. \tag{71}$$

$$\int \sin^{-1} x\, dx = x\, \sin^{-1} x + \sqrt{1-x^2} + C\,. \tag{72}$$

$$\int \cos^{-1} x\, dx = x\, \cos^{-1} x - \sqrt{1-x^2} + C\,. \tag{73}$$

$$\int \tan^{-1} x\, dx = x\, \tan^{-1} x - \tfrac{1}{2} \ln\left(x^2+1\right) + C\,. \tag{74}$$

$$\int (\sin^{-1} x)^2\, dx = (\sin^{-1} x)^2 x - 2x + 2(\sin^{-1} x)\sqrt{1-x^2}\,] + C\,. \tag{75}$$

$$\int \frac{dx}{x\sqrt{x^2+x+1}} = \ln\left|\, \frac{1 + x/2 + \sqrt{x^2+x+1}}{x} \,\right| + C\,. \tag{76}$$

$$\int \frac{1}{x^2 + \sqrt{2}x + 1}\, dx = \sqrt{2}\tan^{-1}[\sqrt{2}x + 1)] + C\,, \tag{77a}$$
$$\int \frac{1}{x^2 - \sqrt{2}x + 1}\, dx = \sqrt{2}\tan^{-1}[\sqrt{2}x - 1)] + C\,. \tag{77b}$$

$$\int \frac{x}{x^2 + \sqrt{2}x + 1}\, dx = \tfrac{1}{2}\ln(x^2 + \sqrt{2}x + 1) - 2\tan^{-1}[\sqrt{2}x + 1] + C\,, \tag{78a}$$
$$\int \frac{x}{x^2 - \sqrt{2}x + 1}\, dx = \tfrac{1}{2}\ln(x^2 - \sqrt{2}x + 1) + 2\tan^{-1}[\sqrt{2}x - 1] + C\,. \tag{78b}$$

3 Virtual Emplacement

So much of mathematics employs tricks that get used over and over in a wide variety of subject areas, yet go unnamed, and thus are hard to explain to one's readers when one uses them. Decades ago I invented the term *virtual emplacement* to refer to the algebraic action of adding a zero to an expression, or multiplying or dividing an expression by unity, or more generally, performing some function and its inverse to an expression, such as

$$x\sqrt{x+y} = \sqrt{x^2(x+y)} \quad \text{when} \quad x, y \geq 0\,. \tag{79}$$

Let's look at the expression $\dfrac{x^2}{1+x^2}$. Can we simplify it?[1] Yes, by performing a virtual emplacement.

$$\frac{x^2}{1+x^2} = \frac{x^2+1-1}{1+x^2} = \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}\,. \tag{80}$$

Stuff like this comes up all the time in integration.

Now, as a real application, consider the integral:

$$\int \frac{x^2}{1+x^2}\,dx = \,? \tag{81}$$

Clearly, we can use the table integral (27), if we can message it into the correct form. But we can do that by employing the result of (80), as well, to get

$$\begin{aligned} \int \frac{x^2}{1+x^2}\,dx &= \int 1\,dx - \int \frac{dx}{1+x^2}\,, \\ \text{hence} \quad \int \frac{x^2}{1+x^2}\,dx &= x - \tan^{-1} x + C\,, \end{aligned} \tag{82}$$

where C is an arbitrary constant.

4 The 'Carrington' (Differential) Equation vs. Integration by Parts

Integration by Parts is one of the most used techniques in the bag of gimmicks of indefinite integration. The technique provides us with an integration identity:

$$\int u\,dv = vu - \int v\,du\,. \tag{83}$$

The proof of it is based on the product rule of differentiation:

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x)\,. \tag{84}$$

Now, we integrate:

$$f(x)g(x) = \int g(x)D_x f(x)\,dx + \int f(x)D_x g(x)\,dx\,, \tag{85}$$

or,

$$f(x)g(x) = \int g(x)\,df + \int f(x)\,dg\,, \tag{86}$$

[1] By 'simplify' in this case, I mean to reduce the given expression to a sum of expressions, each of which is easier to integrate than the original expression.

Now, let $u = f(x)$ and $v = g(x)$, then this last equation becomes

$$uv = \int v\,du + \int u\,dv\,, \tag{87}$$

from which follows (83).

The Carrington[2] differential equation, associated to some integral, is any equation of the form[3]

$$D_x[f(x)g(x)] = [D_x f(x)]g(x) + f(x)D_x g(x) \tag{88}$$

that facilitates an integration problem. The next step is to integrate across (88).

This Carrington equation is not unique, though I will sometimes proffer to the reader 'the Carrington equation', which should be interpreted merely as the *particular* Carrington equation that I chose.

Heuristic: I often set $g(x)$ to be the integrand of the integral and then set $f(x) = x$.

Let's now do an example problem: Find the integral[4]

$$I = \int x \ln x\,dx\,, \tag{89}$$

We start with the Carrington differential equation (which follows the above heuristic):

$$D_x[x^2 \ln x] = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x\,. \tag{90}$$

Now we integrate:

$$\begin{aligned} x^2 \ln x &= 2I + \int x\,dx \\ &= 2I + \frac{1}{2}x^2\,. \end{aligned} \tag{91}$$

From this we get that

$$2I = x^2 \ln x - \frac{1}{2}x^2 + C'\,, \tag{92}$$

or

$$\int x \ln x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C\,. \tag{93}$$

[2]For the moment, this name is a place holder until I can come up with a better name for it. Dr. Carrington refers to a fictional character in the scifi movie *The Thing from Another World*, 1951. This equation is merely an ansatz and is, in most cases, not unique.

[3]The LHS of the Carrington Equation could contain a product of three or more factors, but that gets messy fast.

[4]I will often use the variables I, J, K, etc as placeholders for integrals to reduce the visual mess of the problem.

5 The Integrals

Problem 1:

Find the integral

$$I = \int \tan^5 x \, dx \,. \tag{94}$$

First, noting the identity that $5 = 2 \cdot 2 + 1$, we get that

$$\begin{aligned} \int \tan^5 x \, dx &= \int (\tan^2 x)^2 \tan x \, dx \\ &= \int (\sec^2 x - 1)^2 \tan x \, dx \\ &= \int (\sec^4 x - 2\sec^2 x + 1) \tan x \, dx \\ &= \int (\sec^4 x) \tan x \, dx - 2 \int (\sec^2 x) \tan x \, dx + \int \tan x \, dx \\ &= \int (\sec^3 x) d(\sec x) - 2 \int (\sec x) d(\sec x) + \int \tan x \, dx \\ &= \tfrac{1}{4} \sec^4 x - \sec^2 x + \ln |\sec x| + C \,. \end{aligned} \tag{95}$$

Problem 2:

Find the integral

$$I = \int \sqrt{x^2 + 1} \, dx \,. \tag{96}$$

Let's use a variable substitution:

$$\begin{aligned} x &= \sinh u \,, \\ x^2 + 1 &= \cosh^2 u \,, \\ dx &= \cosh u \, du \,. \end{aligned} \tag{97}$$

Then,

$$\begin{aligned} I(u) &= \int \cosh^2 u \, du \\ &= \tfrac{1}{2} \int [1 + \cosh 2u] \, du \\ &= \tfrac{1}{2} u + \tfrac{1}{4} \int \cosh(2u) \, d(2u) \\ &= \tfrac{1}{2} u + \tfrac{1}{4} \sinh(2u) \\ &= \tfrac{1}{2} u + \tfrac{1}{2} \sinh u \, \cosh u \,. \end{aligned} \tag{98}$$

Hence,

$$I = \int \sqrt{x^2 + 1} \, dx = \tfrac{1}{2} (\sinh^{-1} x + x \sqrt{x^2 + 1}) \,. \tag{99}$$

Problem 3:

Find the integral

$$I = \int \frac{1}{(x^2+1)^2}\, dx\,. \tag{100}$$

Let's try a Carrington equation:

$$D_x\Big[\frac{1}{x}\frac{1}{(x^2+1)}\Big] = \frac{-1}{x^2}\frac{1}{(x^2+1)} - 2\frac{1}{(x^2+1)^2}\,. \tag{101}$$

But

$$\frac{1}{x^2}\frac{1}{(x^2+1)} = \frac{1}{x^2} - \frac{1}{(x^2+1)}\,. \tag{102}$$

So,

$$D_x\Big[\frac{1}{x}\frac{1}{(x^2+1)}\Big] = -\frac{1}{x^2} + \frac{1}{(x^2+1)} - 2\frac{1}{(x^2+1)^2}\,. \tag{103}$$

On integrating, we get

$$\frac{1}{x}\frac{1}{(x^2+1)} = \frac{1}{x} + \tan^{-1}x - 2I + C'\,. \tag{104}$$

Hence,

$$\int \frac{1}{(x^2+1)^2}\, dx = \frac{1}{2}\Big[\frac{x}{(x^2+1)} + \tan^{-1}x\Big] + C\,. \tag{105}$$

Problem 4:

Find the integral

$$I = \int \frac{x\,dx}{(1+2x)^3}\,. \tag{106}$$

Let's use virtual emplacement:

$$\begin{aligned}
\int \frac{x\,dx}{(1+2x)^3} &= \tfrac{1}{2}\int \frac{2x\,dx}{(1+2x)^3} \\
&= \tfrac{1}{2}\int \frac{(1+2x)-1}{(1+2x)^3}\,dx \\
&= \tfrac{1}{2}\int \frac{(1+2x)}{(1+2x)^3}\,dx - \tfrac{1}{2}\int \frac{1}{(1+2x)^3}\,dx \\
&= \tfrac{1}{2}\int \frac{1}{(1+2x)^2}\,dx - \tfrac{1}{2}\int \frac{1}{(1+2x)^3}\,dx \\
&= \tfrac{1}{4}\int \frac{d(1+2x)}{(1+2x)^2} - \tfrac{1}{4}\int \frac{d(1+2x)}{(1+2x)^3} \\
&= -\tfrac{1}{4}\frac{1}{1+2x} + \tfrac{1}{8}\frac{1}{(1+2x)^2} + C \\
&= -\tfrac{1}{8}\Big[\frac{2}{1+2x} - \frac{1}{(1+2x)^2}\Big] + C \\
&= -\tfrac{1}{8}\frac{4x+1}{(1+2x)^2} + C\,.
\end{aligned} \tag{107}$$

Problem 5:

Find the integral

$$I = \int \frac{x \tan^{-1} x}{(1+x^2)^2}\, dx\,. \tag{108}$$

Let's use a Carrington this time:

$$D_x\big[\tan^{-1} x \frac{1}{(1+x^2)}\big] = \frac{1}{(1+x^2)^2} - 2\frac{x\tan^{-1} x}{(1+x^2)^2}\,. \tag{109}$$

On integration, we get,

$$\frac{\tan^{-1} x}{(1+x^2)} = \int \frac{1}{(1+x^2)^2}\, dx - 2I\,. \tag{110}$$

Using the result of Problem 3, we get

$$I = \frac{1}{4}\Big[\frac{x}{(x^2+1)} + \tan^{-1} x - 2\frac{\tan^{-1} x}{(1+x^2)}\Big] + C\,. \tag{111}$$

(Note: WolframAlpha would not integrate this one as a confirmation.)

Problem 6:

Given that

$$\int \frac{dx}{\sinh x} = \ln\Big(\frac{\cosh x - 1}{\sinh x}\Big) + C\,, \tag{112}$$

find the integral

$$I = \int \frac{dx}{x\sqrt{1+x^2}}\,. \tag{113}$$

Let's use a variable substitution:

$$\begin{aligned} x &= \sinh u\,,\\ x^2+1 &= \cosh^2 u\,,\\ dx &= \cosh u\, du\,. \end{aligned} \tag{114}$$

Hence

$$I(u) = \int \frac{du}{\sinh u} = \ln\big(\frac{\cosh u - 1}{\sinh u}\big) + C'\,. \tag{115}$$

Therefore,

$$\int \frac{dx}{x\sqrt{1+x^2}} = \ln\big(\frac{\sqrt{1+x^2}-1}{x}\big) + C\,. \tag{116}$$

Problem 7:

Find the integral

$$I = \int \frac{\ln x}{x^2}\, dx\,. \tag{117}$$

Let's use the Carrington equation:

$$D_x\big[\frac{1}{x}\ln x\big] = -\frac{1}{x^2}\ln x + \frac{1}{x^2}\,. \tag{118}$$

On integrating, we get

$$\frac{1}{x}\ln x = -I - \frac{1}{x} + C\,. \tag{119}$$

On solving for I, we get

$$I = -\frac{1}{x}(\ln x + 1) + C\,. \tag{120}$$

Problem 8:

Find the integral

$$I = \int \left(\frac{\ln x}{x}\right)^2 dx\,. \tag{121}$$

Let's use the Carrington equation:

$$D_x\left[\,\frac{1}{x}(\ln x)^2\,\right] = -\frac{1}{x^2}(\ln x)^2 + \frac{2\ln x}{x^2}\,. \tag{122}$$

On integrating, we get

$$\frac{1}{x}(\ln x)^2 = -I - \frac{2}{x}(\ln x + 1) + C\,, \tag{123}$$

where we used the result of the last problem. On solving for I, we get

$$I = -\frac{1}{x}(\ln x)^2 - \frac{2}{x}(\ln x + 1) + C\,. \tag{124}$$

Problem 9:

Find the integral

$$I = \int \sqrt{a^2 - x^2}\, dx\,. \tag{125}$$

Let's use a variable substitution (refer to Figure 1.):

$$\begin{aligned} x &= a\cos\theta\,,\\ \frac{\sqrt{a^2-x^2}}{a} &= \sin\theta\,,\\ \frac{\sqrt{a^2-x^2}}{x} &= \tan\theta\,,\\ dx &= -a\sin\theta\, d\theta\,. \end{aligned} \tag{126}$$

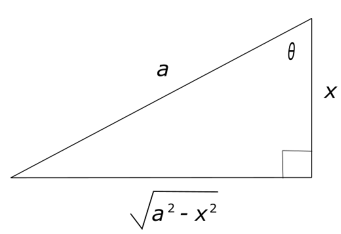

Figure 1. The values on the sides of the triangle are set so that $\cos\theta = x/a$.

Hence,

$$\begin{aligned}I(\theta) &= -a^2 \int \sin^2\theta\, d\theta \\ &= -\frac{a^2}{2}\int [1-\cos(2\theta)]\, d\theta \\ &= -\frac{a^2}{2}\theta + \frac{a^2}{4}\sin(2\theta) + C' \\ &= -\frac{a^2}{2}\theta + \frac{a^2}{24}\sin\theta\cos\theta + C'\,. \end{aligned} \tag{127}$$

So,

$$I(x) = -a^2 \tan^{-1}\frac{\sqrt{a^2-x^2}}{x} + \tfrac{1}{2}x\sqrt{a^2-x^2} + C\,. \tag{128}$$

6 The Weierstrass Substitution

Let

$$t = \tan\left(\frac{x}{2}\right), \tag{129}$$

then

$$\begin{aligned}\sin\left(\frac{x}{2}\right) &= \frac{t}{\sqrt{1+t^2}}\,, \\ \cos\left(\frac{x}{2}\right) &= \frac{1-t^2}{\sqrt{1+t^2}}\,, \\ dx &= \frac{2}{1+t^2}\, dt\,. \end{aligned} \tag{130}$$

For good measure, let's add in:

$$\begin{aligned}\sin x &= \frac{2t}{1+t^2}\,, \\ \cos x &= \frac{1-t^2}{1+t^2}\,, \\ \tan x &= \frac{2t}{1-t^2}\,. \end{aligned} \tag{131}$$

Problem 10:

Using the Weierstrass substitution, find the integral

$$I = \int \frac{1}{1+\sin x}\, dx\,. \tag{132}$$

Then,

$$\begin{aligned}I(t) &= 2\int \frac{dt}{1+t^2+2t} \\ &= 2\int \frac{d(t+1)}{(t+1)^2} \\ &= -2\frac{1}{(t+1)} + C'\,. \end{aligned} \tag{133}$$

Therefore,

$$I(x) = -\frac{2}{\tan\frac{x}{2} + 1} + C\,. \tag{134}$$

7 Problems Left for the Reader:

Reader Problem 1:

To integrate

$$\int x \sin x \cos x \, dx\,, \tag{135}$$

try the Carrington

$$D_x\left[\, x \sin^2 x \,\right]. \tag{136}$$

Reader Problem 2:

Integrate

$$\int \frac{x^4}{1 - x^2}\, dx\,. \tag{137}$$

Hint: Add and subtract unity in the numerator and then factor.

Reader Problem 3:

Integrate

$$\int \sec\, x \, dx\,, \tag{138}$$

by using the Weierstrass substitution.

Reader Problem 4:

Integrate

$$\int \frac{dx}{x^2\sqrt{1 - x^2}}\,. \tag{139}$$

Hint: Try the substitution $x = \sin\theta$.

Reader Problem 5:

Integrate

$$\int \frac{x\, dx}{1 + x^4}\,. \tag{140}$$

Hint: Try the substitution $x^2 = u$.

Reader Problem 6:

Integrate

$$\int \sqrt{\tan\, x}\, dx\,. \tag{141}$$

Hint: Try the substitution $u^2 = \tan\, x$.

Reader Problem 7:

Finish off the integral that I have started below by using a virtual emplacement of unity:

$$\int x^3 \sqrt{x^2+1}\, dx = \tfrac{1}{2} \int x^2 \sqrt{x^2+1}\, d(x^2+1)\,. \tag{142}$$

Reader Problem 8:

Find the integral of

$$\int \sin(\ln x)\, dx \tag{143}$$

by expanding the two Carrington equations (and then integrating them):

$$D_x\big[\, x \cos(\ln x)\,\big] = \cdots\,,$$
$$D_x\big[\, x \sin(\ln x)\,\big] = \cdots\,,$$

and taking linear combinations.

Reader Problem 9:

Integrate

$$\int \frac{dx}{1 - \cos\, x} \tag{144}$$

by multplying both numerator and denominator by $1 + \cos\, x$ and use that $\sin^2 x = 1 - \cos^2 x$.

Reader Problem 10:

Integrate

$$\int x \tan^{-1} x\, dx \tag{145}$$

by employing the Carrington

$$D_x\big[\, x^2 \tan^{-1} x\,\big] = 2x \tan^{-1} x + \frac{x^2}{1+x^2}\,. \tag{146}$$

Reader Problem 11:

Integrate

$$\int x^3 \cos\, x^2\, dx \tag{147}$$

by employing the substitution $u^2 = x$.

Reader Problem 12:

Integrate

$$\int (\tan^3 x + \tan^4 x)\, dx \tag{148}$$

by, first, factoring out a $\tan^2 x$ and then using the identity $\tan^2 x = \sec^2 x - 1$.

Reader Problem 13:

Integrate

$$\int \frac{x^3\,dx}{(x^2+5)^2} \tag{149}$$

by using the Carrington equation

$$D_x\left[x^2\frac{1}{(x^2+5)}\right] = \frac{2x}{(x^2+5)} - 2\frac{x^3}{(x^2+5)^2}\,. \tag{150}$$

Reader Problem 14:

Integrate

$$\int (x+1)\sqrt{x^2+2x}\,dx \tag{151}$$

by completing the square in the radicand.

Reader Problem 15:

Integrate

$$\int \frac{dx}{(x+\sqrt{1+x^2})^2} \tag{152}$$

by using the variable substitution

$$x = \sinh\, u\,, \tag{153}$$

and the indentity

$$\cosh^2 u - \sinh^2\, u = 1 \tag{154}$$

(hint: to be used in the denominator).

8 Appendix:

Show that

$$\sinh z = -i\sin(iz)\,. \tag{155}$$

Using the definitions of the $\sin z$ and $\sinh z$ functions and that $i^2 = -1$:

$$\begin{aligned}\sinh z &= \tfrac{1}{2}(e^z - e^{-z}) \\ &= \tfrac{1}{2}(e^{-i(iz)} - e^{i(iz)}) \\ &= -i\frac{e^{i(iz)} - e^{-i(iz)}}{2i} \\ &= -i\sin(iz)\,. \end{aligned} \tag{156}$$