Equations 1.60a-d on Page 15

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1 Introduction

On page 15 of CAGC [1], we are asked to prove the following relations (1.60a-d), which we relabel here as:

$$
\begin{aligned}
(b \wedge a) \times A &= b \cdot (a \wedge A) - a \cdot (b \wedge A) &\quad (1a)\\
&= b \wedge (a \cdot A) - a \wedge (b \cdot A) &\quad (1b)\\
&= b\, a \cdot A - A \cdot b\, a &\quad (1c)\\
&= b\, a \wedge A - A \wedge b\, a = (ba) \times A\,. &\quad (1d)
\end{aligned}
$$

Some results we can use include:

$$A \times B \equiv \tfrac{1}{2}(AB - BA)\,, \tag{2}$$

and its 'corollaries'

$$c \times B = 0\,, \tag{3}$$

for c a scalar, and the antisymmetry property:

$$B \times A = -A \times B\,. \tag{4}$$

This important expansion identity will be handy:

$$A \times (BC) = (A \times B)C + B(A \times C)\,. \tag{5}$$

Next, are a couple easy identities:

$$
\begin{aligned}
a \cdot A_r &= (-1)^{r+1} A_r \cdot a\,, &\quad (6a)\\
a \wedge A_r &= (-1)^r A_r \wedge a\,. &\quad (6b)
\end{aligned}
$$

From Eq. (1.42) of the text (page 12), we have

$$a \cdot (A_r \wedge B_s) = (a \cdot A_r) \wedge B_s + (-1)^r A_r \wedge (a \cdot\, B_s)\,. \tag{7}$$

From this we get the various corollaries:

$$
\begin{aligned}
b \cdot (a \wedge A_r) &= a \cdot bA_r - a \wedge (b \cdot A_r)\,, &\quad (8a)\\
a \wedge (b \cdot A_r) &= a \cdot bA_r - b \cdot (a \wedge A_r)\,, &\quad (8b)\\
a \cdot (b \wedge A_r) &= a \cdot bA_r - b \wedge (a \cdot A_r)\,, &\quad (8c)\\
b \wedge (a \cdot A_r) &= a \cdot bA_r - a \cdot (b \wedge A_r)\,. &\quad (8d)
\end{aligned}
$$

2 Solution

My approach will be to recast A as just the r-vector A_r and trust to linearity to solve for the more general case of a general multivector A. Besides that, I should point out that my solution seems a bit long, which indicates that there probably exists a shorter proof.

Lemma 1

$$a \cdot (\, b \wedge A_r) = (A_r \wedge b\,) \cdot \, a\,. \tag{9}$$

Proof:

$$\begin{aligned} a \cdot (\, b \wedge A_r) &= (-1)^{r+2} (\, b \wedge A_r) \cdot a \quad \text{(using (6a))} \\ &= (-1)^{2r+2} (A_r \wedge b) \cdot a \\ &= (A_r \wedge b) \cdot a\,. \end{aligned} \tag{10a}$$

■

Lemma 2

$$a \wedge (\, b \cdot A_r) = (A_r \cdot b\,) \wedge \, a\,. \tag{11}$$

Proof: The proof is similar to the proof of the last lemma.

■

Now I can start the proof proper. Let's begin by proving the second result of Eq. (1d)

$$(b \wedge a) \times A_r = (ba) \times A_r\,, \tag{12}$$

where we merely added $b \cdot a$ into the parentheses on the LHS, and noted that since $b \cdot a$ is a scalar, then

$$(b \cdot a) \times A_r = 0\,. \tag{13}$$

Next, we use the antisymmetry of the cross product (4) to get

$$(ba) \times A_r = -A_r \times (ba)\,, \tag{14}$$

So, we can combine this last equation and (12) to get

$$(b \wedge a) \times A_r = -A_r \times (ba)\,. \tag{15}$$

We're now in the position to employ (5):

$$\begin{aligned} (b \wedge a) \times A_r &= -(A_r \times b)a - bA_r \times a \\ &= -(A_r \times b)a - bA_r \times a \\ &= -\tfrac{1}{2}(A_r b - bA_r)a - \tfrac{1}{2}b(A_r a - aA_r) \\ &= -\tfrac{1}{2}A_r ba + \tfrac{1}{2}baA_r \\ &= -\tfrac{1}{2}(A_r \cdot b + A_r \wedge b)a + \tfrac{1}{2}b(a \cdot A_r + a \wedge A_r) \\ &= -\tfrac{1}{2}A_r \cdot b\, a - \tfrac{1}{2}A_r \wedge b\, a + \tfrac{1}{2}b\, a \cdot A_r + \tfrac{1}{2}b\, a \wedge A_r \\ &= -\tfrac{1}{2}(A_r \cdot b) \wedge a - \tfrac{1}{2}(A_r \wedge b) \cdot a + \tfrac{1}{2}b \wedge (a \cdot A_r) + \tfrac{1}{2}b \cdot (a \wedge A_r)\,, \end{aligned} \tag{16}$$

where we attained the last step by recognizing that the $(r-2)$-graded parts and the $(r+2)$-graded parts, respectively, cancelled.

Now we use (9) the replace the second term on the RHS and then change to order to get

$$(b \wedge a) \times A_r = \tfrac{1}{2}b \cdot (a \wedge A_r) + \tfrac{1}{2}b \wedge (a \cdot A_r) - \tfrac{1}{2}a \cdot (b \wedge A_r) - \tfrac{1}{2}(A_r \cdot b) \wedge a\,. \tag{17}$$

So, on our way to prove (1a), we are in some sense half way there, according to the first and third terms on the RHS. What about the second term? According to (8d), we have that

$$b \wedge (a \cdot A_r) = a \cdot b A_r - a \cdot (b \wedge A_r)\,. \tag{18}$$

And from Lemma 2 (11) and (8b), the fourth term becomes

$$(A_r \cdot b) \wedge a = a \wedge (b \cdot A_r) = a \cdot b A_r - b \cdot (a \wedge A_r)\,. \tag{19}$$

On taking half of (18) and subtracting from it half of (19), we get

$$\tfrac{1}{2} b \wedge (a \cdot A_r) - \tfrac{1}{2}(A_r \cdot b) \wedge a = \tfrac{1}{2} b \cdot (a \wedge A_r) - \tfrac{1}{2} a \cdot (b \wedge A_r)\,. \tag{20}$$

Substituting this into (17), we get

$$(b \wedge a) \times A_r = b \cdot (a \wedge A_r) - a \cdot (b \wedge A_r)\,, \tag{21}$$

which is the text Eq. (1.60a) or my Eq. (1a).

Now, to arrive at the text Eq. (1.60b) or my Eq. (1b), just substitute into the last equation from Equations (8a) and (8c).

Lemma 3

$$A_r \cdot (b \wedge a) = (A_r \cdot b) \cdot a\,. \tag{22}$$

Proof:

$$\begin{aligned} A_r \cdot (b \wedge a) &= \langle A_r \cdot (b \wedge a) \rangle_{r-2} \\ &= \langle A_r (b \wedge a) \rangle_{r-2} \\ &= \langle A_r b\, a \,\rangle_{r-2} \\ &= \langle (A_r \cdot b)\, a \,\rangle_{r-2} \\ &= (A_r \cdot b) \cdot a\,. \end{aligned} \tag{23}$$

■

Lemma 4

$$b \cdot (a \cdot A_r) = (A_r \cdot b) \cdot a\,. \tag{24}$$

Proof:

$$\begin{aligned} b \cdot (a \cdot A_r) &= \langle (\, b \wedge a) \cdot A_r \rangle_{r-2} \\ &= \langle (\, b \wedge a) A_r \rangle_{r-2} \\ &= \left[\langle \left((\, b \wedge a) A_r \right)^{\dagger} \rangle_{r-2} \right]^{\dagger} \\ &= (-1)^{(r-2)(r-3)/2} \langle A_r^{\dagger} (\, b \wedge a)^{\dagger} \rangle_{r-2} \\ &= -(-1)^{(r-2)(r-3)/2} (-1)^{(r)(r-1)/2} \langle A_r (\, b \wedge a) \rangle_{r-2} \\ &= \langle A_r (\, b \wedge a) \rangle_{r-2} \\ &= A_r \cdot (\, b \wedge a) \\ &= (A_r \cdot b) \cdot a\,. \end{aligned}$$

■

Now we're ready to show (1c).

$$\begin{aligned}(b \wedge a) \times A_r &= b \wedge (a \cdot A_r) - a \wedge (b \cdot A_r) \\ &= b \wedge (a \cdot A_r) - (A_r \cdot b) \wedge a\,,\end{aligned} \tag{25}$$

where we used (11).

Because of (24), we can write

$$0 = b \cdot (a \cdot A_r) - (A_r \cdot b) \cdot a\,. \tag{26}$$

Adding this to (25), we get

$$(b \wedge a) \times A_r = b\,(a \cdot A_r) - (A_r \cdot b)\,a\,, \tag{27}$$

which is Eq. (1c).

Our last proof is upon us.

$$\begin{aligned}(b \wedge a) \times A &= b \cdot (a \wedge A) - a \cdot (b \wedge A) \\ &= b \cdot (a \wedge A) - (A_r \wedge b\,) \cdot\, a\,,\end{aligned} \tag{28}$$

where we used Lemma 1. Now we add

$$0 = b \wedge (a \wedge A) - (A_r \wedge b\,) \wedge\, a \tag{29}$$

to the last equation to get

$$(b \wedge a) \times A = b\,a \wedge A - A \wedge b\,a\,. \tag{30}$$

And we are finished.

References

[1] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, D. Reidel Publishing Co. (Kluwer Academic Publishers), 1987.

[2] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.