Problem 7.2 on Page 38

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1 Problem

On page 38 of NFCM [1], we find Problem (7.2).

Let

$$A = \alpha + \mathbf{a}\,, \tag{1}$$

where α is a scalar and $\mathbf{a}$ is a nonzero vector.

(a) Find A^{-1} and determine when this entity does not exist.

$$A^{-1} = \frac{1}{\alpha + \mathbf{a}} = \frac{\alpha - \mathbf{a}}{(\alpha + \mathbf{a})(\alpha - \mathbf{a})} = \frac{\alpha - \mathbf{a}}{\alpha^2 - \mathbf{a}^2}\,, \quad \text{where } \alpha \neq \pm|\,\mathbf{a}\,|\,. \tag{2}$$

In the case $\alpha = \pm|\,\mathbf{a}\,|$ (or $|\,\mathbf{a}\,| = \pm\alpha$), (1) takes the form

$$A_{\pm} = \alpha(1 \pm \hat{\mathbf{a}})\,. \tag{3}$$

(b) If A does not have an inverse, show that it can be normalized so that

$$A^2 = A\,. \tag{4}$$

Substituting (3) into this, we have that

$$\alpha^2(1 \pm \hat{\mathbf{a}})^2 = \alpha(1 \pm \hat{\mathbf{a}})\,. \tag{5}$$

From the scalar part of this equation we get that $2\alpha^2 = \alpha$, and since $\alpha \neq 0$ then $\alpha = 1/2$. Therefore,

$$A_{\pm} = \tfrac{1}{2}(1 \pm \hat{\mathbf{a}})\,. \tag{6}$$

Definition: A number $\mathbf{e}$ satisfying the equation

$$\mathbf{e}^2 = \mathbf{e} \tag{7}$$

is said to be *idempotent*. If the solution to (7) is either 0 or 1, the solution is said to be *trivial*, otherwise, it is said to be *nontrivial*. We have just shown that nontrivial idempotents are not invertible.

(c) Show that the product of an invertible multivector A with a nontrivial idempotent $\mathbf{e}$ is not invertible. Proof: Let B be a multivector that can be factored as

$$B = A\mathbf{e}\,. \tag{8}$$

Assuming that B has an inverse, it will have the form

$$B^{-1} = \mathbf{e}^{-1} A^{-1} \,, \tag{9}$$

but this is not possible since $\mathbf{e}$ is not invertible. The factorization of $B = \mathbf{e}A$ leads to a similar proof and won't be covered here.

(d) Find an idempotent that does have an inverse.

Obviously, an invertible idempotent must be trivial, if it exists at all. And the only suitable candidate turns out to be unity.

References

[1] D. Hestenes, *New Foundations in Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.