Product of Diagonals Area

P. Reany

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1 The Problem

It is well-known in plane geometry that the area of a rhombus can be calculated by taking half the product of its diagonals. That is,

$$\text{Area} = \tfrac{1}{2} d_1 d_2 \,, \tag{1}$$

as depicted in the following figure:

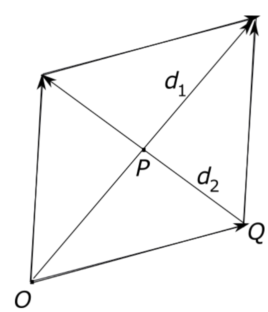

Figure 1. This is the problem as set up according to plane geometry. The lengths of the diagonals of the rhombus are represented by the numbers d_1 and d_2.

Our task here is to prove this result using geometric algebra.

2 The Solution

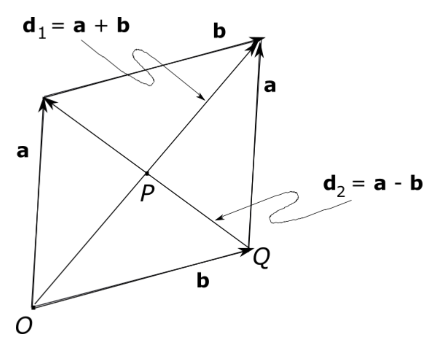

Figure 2. We've constructed a parallelogram with all sides of equal lengths, i.e., a rhombus. We will prove that the diagonals, which intersect at point P, interesect each other at right angles. This result can also be proven vectorially by showing that $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 0$.

Lemma: The diagonals of a rhombus intersect each other at right angles.

Proof: First, we note that the algebraic depiction of the claim that the sides of the rhombus have equal lengths requires that

$$|\,\mathbf{a}\,| = |\,\mathbf{b}\,|\,, \tag{2}$$

or, $a = b$. Now, to prove that the diagonals intersect in right angles, we have that

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = a^2 + \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - b^2 = 0\,. \tag{3}$$

Hence, the two diagonals intersect each other at right angles.

Main Theorem: According to this theorem, the area A of a rhombus is given as

$$A = \tfrac{1}{2} d_1 d_2\,. \tag{4}$$

First Proof: In this proof we'll work solely with areas represented by bivectors. In this case, the area of the rhombus is simply

$$\mathbf{A} = \mathbf{a} \wedge \mathbf{b}\,. \tag{5}$$

On taking the absolute values of this, we get that

$$A = ab \sin\theta = a^2 \sin\theta\,, \tag{6}$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. So, can we arrive at (5) by using the vector diagonals by taking $\frac{1}{2}\mathbf{d}_1 \wedge \mathbf{d}_2$? Let's see.

$$\begin{aligned} \tfrac{1}{2}\mathbf{d}_1 \wedge \mathbf{d}_2 &= \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} - \mathbf{b}) \\ &= \tfrac{1}{2}(-\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a}) \\ &= -\mathbf{a} \wedge \mathbf{b}\,. \end{aligned} \tag{7}$$

This is the correct result, ignoring the minus sign. The reason the minus sign appeared is because we have been careless in setting-up the wedge products. The bivector $\mathbf{a} \wedge \mathbf{b}$ has positive (counterclockwise) orientation, whereas the bivector $\mathbf{d}_1 \wedge \mathbf{d}_2$ has negative (clockwise) orientation.

Second Proof: This proof is similar to the last proof, but this time we'll work only with the geometric product. Since the inner product of $\mathbf{d}_1$ and $\mathbf{d}_2$ is zero, we may write

$$\begin{aligned} \tfrac{1}{2}\mathbf{d}_1 \wedge \mathbf{d}_2 &= \tfrac{1}{2}\mathbf{d}_1\mathbf{d}_2 \\ &= \tfrac{1}{2}(\mathbf{a} + \mathbf{b})(\mathbf{a} - \mathbf{b}) \\ &= \tfrac{1}{2}(\mathbf{a}^2 - \mathbf{ab} + \mathbf{ba} - \mathbf{b}^2) \\ &= \tfrac{1}{2}(-\mathbf{ab} + \mathbf{ba}) \qquad \text{(Why?)} \\ &= -\mathbf{a} \wedge \mathbf{b}\,, \end{aligned} \tag{8}$$

which is the same result as last time.

Third Proof: This time, we'll work directly with the scalars d_1 and d_2, though it will be more convenient to work with their squares. As before, let θ be the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \cos\theta$, and

$$\begin{aligned} d_1^2 = \mathbf{d}_1^2 = |\,\mathbf{a} + \mathbf{b}\,|^2 = a^2 + 2\mathbf{a} \cdot \mathbf{b} + b^2 = 2a^2(1 + \cos\theta)\,, \\ d_2^2 = \mathbf{d}_2^2 = |\,\mathbf{a} - \mathbf{b}\,|^2 = a^2 - 2\mathbf{a} \cdot \mathbf{b} + b^2 = 2a^2(1 - \cos\theta)\,. \end{aligned} \tag{9}$$

Therefore,

$$\tfrac{1}{4}d_1^2 d_2^2 = a^2(1+\cos\theta)\,a^2(1-\cos\theta) = a^4(1-\cos^2\theta) = a^4\sin^2\theta\,. \tag{10}$$

On taking the square root of this, we get

$$\tfrac{1}{2}d_1 d_2 = a^2\sin\theta\,. \tag{11}$$

But

$$A^2 = |\,\mathbf{a}\wedge\mathbf{b}\,|^2 = a^2b^2\sin^2\theta = a^4\sin^2\theta\,, \tag{12}$$

as we computed before. On taking the square root of this and comparing to (11), we have that

$$A = \tfrac{1}{2}d_1 d_2\,. \tag{13}$$

References

[1] D. Hestenes, *New Foundations in Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.