Problem 1.10 on Page 47

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1 Introduction

On page 47 of NFCM [1], Problem (1.10), we are asked to prove the following relation

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{C}_r)+\mathbf{a}\wedge(\mathbf{a}\cdot\mathbf{C}_r)=\mathbf{a}\cdot\mathbf{b}\,\mathbf{C}_r\,. \tag{1}$$

2 Solution

My approach will be to recast both terms on the LHS in terms of geometric products only and then add them together. In doing this, I'll make good use of the two identities

$$\mathbf{a}\cdot\mathbf{A}_r=\tfrac{1}{2}(\mathbf{a}\mathbf{A}_r-(-1)^r\mathbf{A}_r\mathbf{a})\,, \tag{2a}$$

$$\mathbf{a}\wedge\mathbf{A}_r=\tfrac{1}{2}(\mathbf{a}\mathbf{A}_r+(-1)^r\mathbf{A}_r\mathbf{a})\,. \tag{2b}$$

Now, to recast the first term on the LHS:[1]

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{C}_r)&=\tfrac{1}{2}\big[\,\mathbf{a}(\mathbf{b}\wedge\mathbf{C}_r)-(-1)^{r+1}(\mathbf{b}\wedge\mathbf{C}_r)\mathbf{a}\,\big]\\&=\tfrac{1}{4}\big[\,\mathbf{a}(\mathbf{b}\mathbf{C}_r+(-1)^r\mathbf{C}_r\mathbf{b})-(-1)^{r+1}(\mathbf{b}\mathbf{C}_r+(-1)^r\mathbf{C}_r\mathbf{b})\mathbf{a}\,\big]\\&=\tfrac{1}{4}\big[\,\mathbf{a}\mathbf{b}\mathbf{C}_r+(-1)^r\mathbf{a}\mathbf{C}_r\mathbf{b}+(-1)^r\mathbf{b}\mathbf{C}_r\mathbf{a}+\mathbf{C}_r\mathbf{b}\mathbf{a}\,\big]\,.\end{aligned} \tag{3}$$

Next, to recast the second term on the LHS:

$$\begin{aligned}\mathbf{b}\wedge(\mathbf{a}\cdot\mathbf{C}_r)&=\tfrac{1}{2}\big[\,\mathbf{b}(\mathbf{a}\cdot\mathbf{C}_r+(-1)^{r-1}(\mathbf{a}\cdot\mathbf{C}_r)\mathbf{b}\,\big]\\&=\tfrac{1}{4}\big[\,\mathbf{b}(\mathbf{a}\mathbf{C}_r-(-1)^r\mathbf{C}_r\mathbf{a})+(-1)^{r-1}\mathbf{a}\mathbf{C}_r-(-1)^r\mathbf{C}_r\mathbf{a})\mathbf{b}\,\big]\\&=\tfrac{1}{4}\big[\,\mathbf{b}\mathbf{a}\mathbf{C}_r-(-1)^r\mathbf{b}\mathbf{C}_r\mathbf{a}+(-1)^{r-1}\mathbf{a}\mathbf{C}_r\mathbf{b}+\mathbf{C}_r\mathbf{a}\mathbf{b}\,\big]\end{aligned} \tag{4}$$

After we add both terms on the LHS of (1) and then perform a lot of cancellations, we get

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{C}_r)+\mathbf{a}\wedge(\mathbf{a}\cdot\mathbf{C}_r)&=\tfrac{1}{4}\big[\,(\mathbf{a}\mathbf{b}+\mathbf{b}\mathbf{a})\mathbf{C}_r+\mathbf{C}_r(\mathbf{a}\mathbf{b}+\mathbf{b}\mathbf{a})\,\big]\\&=\mathbf{a}\cdot\mathbf{b}\,\mathbf{C}_r\,,\end{aligned} \tag{5}$$

which is what we were to show.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.

[1] It's important to recast these dot and wedge products from the outside in.