Problem 1.4 on Page 47

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1 Introduction

On page 47 of NFCM [1], we find the equation

$$\alpha \mathbf{x} + \mathbf{x} \cdot \mathbf{B} = \mathbf{a}\,, \tag{1}$$

where $\mathbf{a}$ is a vector, $\mathbf{B}$ is a bivector, and α is not zero. Our job is to solve this equation for $\mathbf{x}$.

2 Solution

First, I want to make a couple useful substitutions:

$$\mathbf{b} = \mathbf{a}\alpha^{-1}\,, \tag{2}$$

$$\mathbf{A} = \mathbf{B}\alpha^{-1}\,. \tag{3}$$

Then Eq. (1) becomes

$$\mathbf{x} + \mathbf{x} \cdot \mathbf{A} = \mathbf{b}\,. \tag{4}$$

Lemma:

$$(\mathbf{x} \cdot \mathbf{A}) \wedge \mathbf{A} = 0\,. \tag{5}$$

Now,

$$\begin{aligned} \langle \mathbf{x} \rangle_3 &= \langle \mathbf{x}\mathbf{A}\mathbf{A}^{-1} \rangle_3 \\ &= \langle (\mathbf{x} \cdot \mathbf{A})\mathbf{A}^{-1} \rangle_3 + \langle (\mathbf{x} \wedge \mathbf{A}) \cdot \mathbf{A}^{-1} \rangle_3 + \langle (\mathbf{x} \wedge \mathbf{A}) \wedge \mathbf{A}^{-1} \rangle_3 \\ &= \langle (\mathbf{x} \cdot \mathbf{A})\mathbf{A}^{-1} \rangle_3 \\ &= (\mathbf{x} \cdot \mathbf{A}) \wedge \mathbf{A}^{-1}\,, \end{aligned} \tag{6}$$

where all the other terms went to zero by the grade selector alone. But $\langle \mathbf{x} \rangle_3 = 0$. Therefore $(\mathbf{x} \cdot \mathbf{A}) \wedge \mathbf{A}^{-1} = 0$. And, since $(\mathbf{x} \cdot \mathbf{A}) \wedge \mathbf{A}^{-1}$ differs from $(\mathbf{x} \cdot \mathbf{A}) \wedge \mathbf{A}$ by only a nonzero scalar factor, then that establishes (5).

Now we can wedge through by $\mathbf{A}$ on the right sides of (4), and then use (5) to get

$$\mathbf{x} \wedge \mathbf{A} = \mathbf{b} \wedge \mathbf{A}\,. \tag{7}$$

At this point, we split the geometric product and substitute our results:

$$\begin{aligned} \mathbf{x}\mathbf{A} &= \mathbf{x} \cdot \mathbf{A} + \mathbf{x} \wedge \mathbf{A} \\ &= (\mathbf{b} - \mathbf{x}) + \mathbf{b} \wedge \mathbf{A}\,. \end{aligned} \tag{8}$$

Collecting $\mathbf{x}$ on the LHS, yields

$$\mathbf{x}(1+\mathbf{A}) = \mathbf{b} + \mathbf{b} \wedge \mathbf{A}\,. \tag{9}$$

Multiplying through by $(1-\mathbf{A})$ on the right and then simplifying, we get

$$\begin{aligned} \mathbf{x} &= \frac{(\mathbf{b} + \mathbf{b} \wedge \mathbf{A})(1-\mathbf{A})}{1+|\,\mathbf{A}\,|^2} \\ &= \frac{\mathbf{b} - \mathbf{b} \cdot \mathbf{A} - \mathbf{b} \wedge \mathbf{A}\mathbf{A}}{1+|\,\mathbf{A}\,|^2}\,. \end{aligned} \tag{10}$$

On substituting back in the original variables, we have

$$\mathbf{x} = \frac{\alpha^2\mathbf{a} + \alpha\mathbf{B} \cdot \mathbf{a} - \mathbf{a} \wedge \mathbf{B}\mathbf{B}}{\alpha(\alpha^2 + |\,\mathbf{B}\,|^2)}\,, \tag{11}$$

where we have used that $\mathbf{a} \cdot \mathbf{B} = -\mathbf{B} \cdot \mathbf{a}$. Let's prove this.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{B} &= \langle \mathbf{a}\mathbf{B} \rangle_1 \\ &= \langle \mathbf{a}\mathbf{B} \rangle_1^\dagger \\ &= -\langle \mathbf{B}\mathbf{a} \rangle_1 \\ &= -\mathbf{B} \cdot \mathbf{a}\,. \end{aligned} \tag{12}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.