Problem 1.6b on Page 47

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1 Introduction

On page 47 of NFCM [1], we need to prove the following relation

$$\langle \mathbf{a} \wedge \mathbf{bc} \wedge \mathbf{b} \rangle_2 = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{b} \,. \tag{1}$$

2 Solution

My approach will be to expand both so that they will meet in the middle.

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{b} &= \mathbf{b} \cdot (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \\ &= \mathbf{b} \cdot \mathbf{a}\,\mathbf{b} \wedge \mathbf{c} - \mathbf{b}^2\,\mathbf{a} \wedge \mathbf{c} + \mathbf{b} \cdot \mathbf{c}\,\mathbf{a} \wedge \mathbf{b} \,. \end{aligned} \tag{2}$$

Now,

$$\begin{aligned} \langle \mathbf{a} \wedge \mathbf{bc} \wedge \mathbf{b} \rangle_2 &= \langle (\mathbf{ab} - \mathbf{a} \cdot \mathbf{b})\mathbf{c} \wedge \mathbf{b} \rangle_2 \\ &= \langle (\mathbf{ab})\mathbf{c} \wedge \mathbf{b} \rangle_2 - \mathbf{a} \cdot \mathbf{b}\,\mathbf{c} \wedge \mathbf{b} \\ &= \langle \mathbf{a}(\mathbf{bc} \wedge \mathbf{b}) \rangle_2 - \mathbf{a} \cdot \mathbf{b}\,\mathbf{c} \wedge \mathbf{b} \\ &= \langle \mathbf{a}(\mathbf{b} \cdot \mathbf{c} \wedge \mathbf{b}) \rangle_2 - \mathbf{a} \cdot \mathbf{b}\,\mathbf{c} \wedge \mathbf{b} \\ &= \langle \mathbf{a}(\mathbf{b} \cdot \mathbf{c}\,\mathbf{b} - \mathbf{c}\,\mathbf{b}^2) \rangle_2 - \mathbf{a} \cdot \mathbf{b}\,\mathbf{c} \wedge \mathbf{b} \\ &= \mathbf{b} \cdot \mathbf{c}\,\mathbf{a} \wedge \mathbf{b} - \mathbf{b}^2\,\mathbf{a} \wedge \mathbf{c} - \mathbf{a} \cdot \mathbf{b}\,\mathbf{c} \wedge \mathbf{b} \\ &= \mathbf{b} \cdot \mathbf{c}\,\mathbf{a} \wedge \mathbf{b} - \mathbf{b}^2\,\mathbf{a} \wedge \mathbf{c} + \mathbf{b} \cdot \mathbf{a}\,\mathbf{b} \wedge \mathbf{c} \,. \end{aligned} \tag{3}$$

Since this last result is the same as the last result of (2) then have demonstrated what we needed to show.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.