Problem 1.13 on Page 48

P. Reany

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1 Introduction

On page 48 of NFCM [1], we find Problem (1.13):
Prove that

1) $\langle A^\dagger \rangle_r = \langle A \rangle_r^\dagger = (-1)^{r(r-1)/2} \langle A \rangle_r$,
2) $\langle AB \rangle_r = (-1)^{r(r-1)/2} \langle B^\dagger A^\dagger \rangle_r$.

2 Solutions

Part 1: First, let A be expressed as the sum of its graded parts.

$$A = \sum_{i=0}^{n} A_i \,. \tag{1}$$

Then,

$$\langle A^\dagger \rangle_r = \langle \big(\sum_{i=0}^{n} A_i \,\big)^\dagger \rangle_r = \langle \sum_{i=0}^{n} A_i^\dagger \,\rangle_r = A_r^\dagger \,, \tag{2}$$

whereas

$$\langle A \rangle_r^\dagger = \big(\langle \sum_{i=0}^{n} A_i \,\rangle_r \big)^\dagger = \big(A_r \big)^\dagger = A_r^\dagger \,. \tag{3}$$

Hence,

$$\langle A^\dagger \rangle_r = \langle A \rangle_r^\dagger \,. \tag{4}$$

Now, to show that $\langle A \rangle_r^\dagger = (-1)^{r(r-1)/2} \langle A \rangle_r$, I need only show that

$$A_r^\dagger = (-1)^{r(r-1)/2} A_r \,. \tag{5}$$

Proof by Induction:
Base Case: $r = 1$: $A_1^\dagger = (-1)^{1(0)/2} A_1 = A_1$. ✓

Now we assume the result is true for case r and use that to prove that it is true for case $r+1$: First, A_{r+1} has the form $a \wedge A_r$, where a is a vector: [1]

$$\begin{aligned} A_{r+1}^\dagger &= (a \wedge A_r)^\dagger \\ &= A_r^\dagger \wedge a^\dagger \\ &= (-1)^{r(r-1)/2} A_r \wedge a \\ . \end{aligned} \tag{6}$$

[1] I could have chosen the form $A_r \wedge a$ because it too would give us $r+1$ vectors wedged together into an $(r+1)$-vector.

Now, to bring the factor a back to the left of the factor A_r, we need to perform r transpositions with the factors of A_r, leaving us with the additional corrective coefficient factor of $(-1)^r$; hence

$$\begin{aligned} A_{r+1}^{\dagger} &= (-1)^{r(r-1)/2}(-1)^r a \wedge A_r \\ &= (-1)^{\frac{r(r-1)+2r}{2}} a \wedge A_r \\ &= (-1)^{\frac{(r+1)r}{2}} a \wedge A_r \\ &= (-1)^{\frac{(r+1)r}{2}} A_{r+1} \,. \qquad \checkmark \end{aligned} \tag{7}$$

Part 2: I consider this as a corollary of the previous case, as most of the heavy lifting has already been done by it.

I'm going to rewrite the result of the first part as

$$\langle C \rangle_r^{\dagger} = (-1)^{r(r-1)/2} \langle C \rangle_r \,. \tag{8}$$

Now, since $\langle C \rangle_r = (\langle C \rangle_r^{\dagger})^{\dagger}$, then, taking the dagger operator on both sides of the last equation and substituting AB in for C, we have

$$\begin{aligned} \langle AB \rangle_r &= (-1)^{r(r-1)/2} \langle AB \rangle_r^{\dagger} \\ &= (-1)^{r(r-1)/2} \langle (AB)^{\dagger} \rangle_r \\ &= (-1)^{r(r-1)/2} \langle B^{\dagger} A^{\dagger} \rangle_r \end{aligned} \tag{9}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.