Problem 3.3 on Page 62

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1 Solutions

On page 62 of NFCM [1], we find Problem (3.3) in four parts:

Part 1: Prove that $\mathbf{a}\cdot\mathbf{b} = -i[\mathbf{a}\wedge(i\mathbf{b})]$.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \langle\mathbf{ab}\rangle\\ &= -i^2\langle\mathbf{ab}\rangle\\ &= -i\langle\mathbf{a}(i\mathbf{b})\rangle_3\\ &= -i\langle\mathbf{a}\wedge(i\mathbf{b})\rangle_3\\ &= -i[\,\mathbf{a}\wedge(i\mathbf{b})\,]\,. \end{aligned} \tag{1}$$

Part 2: Prove that $\mathbf{b}\times\mathbf{a} \equiv i(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}\cdot(i\mathbf{b}) = -\mathbf{b}\cdot(i\mathbf{a})$.

We can start with

$$\begin{aligned}&\langle i\mathbf{ab}\rangle_1 = \langle\mathbf{a}i\mathbf{b}\rangle_1 \quad \text{or}\\ &i\mathbf{a}\wedge\mathbf{b} = \mathbf{a}\cdot(i\mathbf{b})\,.\end{aligned} \tag{2}$$

To get the last result, since the reverse of a vector leaves the vector unchanged, and since $i^\dagger = -i$,

$$\langle\mathbf{a}i\mathbf{b}\rangle_1 = \langle\mathbf{a}i\mathbf{b}\rangle_1^\dagger = -\langle\mathbf{b}i\mathbf{a}\rangle_1\,. \tag{3}$$

Therefore,

$$\mathbf{a}\cdot(i\mathbf{b}) = -\mathbf{b}\cdot(i\mathbf{a})\,. \tag{4}$$

Part 3: Prove that $\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) = -\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}\,\mathbf{c} - \mathbf{a}\cdot\mathbf{b}\,\mathbf{c}$.

First, we already know that $\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}\,\mathbf{c} - \mathbf{a}\cdot\mathbf{b}\,\mathbf{c}$. So, that leaves us to show that $-\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})$. With $\mathbf{a}\times\mathbf{b} = -i\mathbf{a}\wedge\mathbf{b}$, we write

$$\begin{aligned}-\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= i\mathbf{a}\wedge(\mathbf{b}\times\mathbf{c})\\ &= i\mathbf{a}\wedge(-i\mathbf{b}\wedge\mathbf{c})\\ &= \tfrac{1}{2}i[\,\mathbf{a}(-i\mathbf{b}\wedge\mathbf{c}) - (-i\mathbf{b}\wedge\mathbf{c})\mathbf{a}\,]\\ &= \tfrac{1}{2}[\,\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\wedge\mathbf{c})\mathbf{a}\,]\\ &= \tfrac{1}{2}[\,\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\wedge\mathbf{c})\cdot\mathbf{a}\,]\\ &= \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})\,.\end{aligned} \tag{5}$$

Part 4: Prove that $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = i\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \frac{1}{2}(\mathbf{abc} - \mathbf{cba})$.

We'll start with the RHS:

$$\begin{aligned} \tfrac{1}{2}(\mathbf{abc} - \mathbf{cba}) &= \tfrac{1}{2}[\,\mathbf{a}(\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \wedge \mathbf{c}) - (\mathbf{c} \cdot \mathbf{b} + \mathbf{c} \wedge \mathbf{b})\mathbf{a}\,] \\ &= \tfrac{1}{2}[\,\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) + (\mathbf{b} \wedge \mathbf{c})\mathbf{a}\,] \\ &= \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})\,. \end{aligned} \tag{6}$$

Now, we just need to connect to the middle term.

$$\begin{aligned} i\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= i\tfrac{1}{2}(\mathbf{a}(\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})\mathbf{a}\,] \\ &= \tfrac{1}{2}[\,\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) + (\mathbf{b} \wedge \mathbf{c})\mathbf{a}\,] \\ &= \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})\,. \end{aligned} \tag{7}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.