Problem 4.5 on Page 71

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1 The Problem

On page 71 of NFCM [1], we find problem (4.5): Given **a** and **b** as unit vectors and that

$$\mathbf{ab} = e^{\mathbf{i}\theta}\,, \tag{1}$$

show that

(a) $(\mathbf{ab})^\dagger = \mathbf{ba} = e^{-\mathbf{i}\theta}\,,$

(b) $(\mathbf{a}-\mathbf{b})^2 = 4\sin^2\frac{1}{2}\theta\,,$

(c) $(\mathbf{a}+\mathbf{b})^2 = 4\cos^2\frac{1}{2}\theta\,.$

2 Solution to Part (a)

We begin with the observing that $\mathbf{i}^\dagger = -\mathbf{i}$. Then, taking the reverse operation on the LHS of (1):

$$(\mathbf{ab})^\dagger = \mathbf{b}^\dagger\mathbf{a}^\dagger = \mathbf{ba}\,. \tag{2}$$

Next, we take the reverse operation on the RHS of (1):

$$(e^{\mathbf{i}\theta})^\dagger = (\cos\theta + \mathbf{i}\sin\theta)^\dagger = \cos\theta - \mathbf{i}\sin\theta = e^{-\mathbf{i}\theta}\,, \tag{3}$$

and we have established Part (a).

For Parts (b) and (c) we will make use of the following figure.

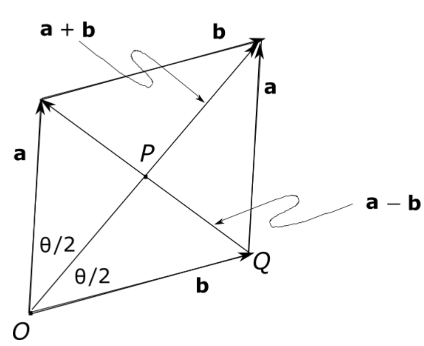

Figure 1. We've constructed a parallelogram with all sides equal, hence, a rhombus. By symmetry, we can prove that the diagonals, which intersect at point P, bisect each other, and have right angles all around. This latter result can also be proven algebraically by showing that $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = 0$.

First, let's prove that the two angles at point O are actually equal by using vector methods. Let $\mathbf{u}$ be a unit vector along the vector $\mathbf{a}+\mathbf{b}$. Then we need to show that

$$\mathbf{a}\cdot\mathbf{u} = \mathbf{b}\cdot\mathbf{u}\,. \tag{4}$$

But

$$\mathbf{u} = \frac{\mathbf{a}+\mathbf{b}}{|\,\mathbf{a}+\mathbf{b}\,|}\,. \tag{5}$$

So, (4) becomes

$$\mathbf{a}\cdot\frac{\mathbf{a}+\mathbf{b}}{|\,\mathbf{a}+\mathbf{b}\,|} \;\stackrel{?}{=}\; \mathbf{b}\cdot\frac{\mathbf{a}+\mathbf{b}}{|\,\mathbf{a}+\mathbf{b}\,|}\,. \tag{6}$$

On multiplying through by $|\,\mathbf{a}+\mathbf{b}\,|$, we have

$$\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) \;\stackrel{?}{=}\; \mathbf{b}\cdot(\mathbf{a}+\mathbf{b})\,. \tag{7}$$

On expanding this and simplifying, we see that the equality holds. What this means is that the angle between $\mathbf{a}$ and the long diagonal is the same measure as that between $\mathbf{b}$ and the long diagonal; hence, the two angles are equal, and equal to $\theta/2$.

3 Solution to Part (b)

Next, we are to show that $(\mathbf{a}-\mathbf{b})^2 = 4\sin^2\frac{1}{2}\theta$. We'll need only trigonometry at this point. Referring to Fig. 1, from the right triangle ΔOPQ, we have that

$$\sin\tfrac{1}{2}\theta = \frac{\frac{1}{2}|\,\mathbf{a}-\mathbf{b}\,|}{b} = \tfrac{1}{2}|\,\mathbf{a}-\mathbf{b}\,|\,. \tag{8}$$

On squaring this and multiplying through by 4, we get

$$(\mathbf{a}-\mathbf{b})^2 = 4\sin^2\tfrac{1}{2}\theta\,, \tag{9}$$

which is what we were to show.

4 Solution to Part (c)

This next proof is similar to the last one. This time we take the cosine of the angle, to get

$$\cos\tfrac{1}{2}\theta = \frac{\frac{1}{2}|\,\mathbf{a}+\mathbf{b}\,|}{b} = \tfrac{1}{2}|\,\mathbf{a}+\mathbf{b}\,|\,. \tag{10}$$

On squaring this and multiplying through by 4, we get

$$(\mathbf{a}+\mathbf{b})^2 = 4\cos^2\tfrac{1}{2}\theta\,, \tag{11}$$

which is what we were to show.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.