Problem 5.6a on Page 78

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1 The Problem

On page 78 of NFCM [1], we find problem (5.6a): Given that $I^2 = -1$ and that I, A, and B mutually commute, show that

$$\cos{(A + IB)} = \cos A \, \cosh B - I \sinh B \, \sin A \, . \tag{1}$$

A few identities we will need are

$$e^X = \cosh X + \sinh X \, , \tag{2a}$$
$$2 \sinh X = e^X - e^{-X} \, , \tag{2b}$$
$$2 \cos X = e^{IX} + e^{-IX} \, . \tag{2c}$$

Now,

$$\begin{aligned}
2\cos{(A + IB)} &= e^{I(A+IB)} + e^{-I(A+IB)} \\
&= e^{IA}e^{-B} + e^{-IA}e^{B} \\
&= e^{IA}e^{-B} + e^{-IA}e^{-B} - e^{-IA}e^{-B} + e^{-IA}e^{B} \quad \text{(add in and subtract out)} \\
&= (e^{IA} + e^{-IA})e^{-B} + e^{-IA}(e^{B} - e^{-B}) \\
&= 2\cos A\,(\cosh B - \sinh B) + 2(\cos A - I \sin A) \sinh B \\
&= 2(\cos A \, \cosh B - I \sinh B \, \sin A) \, .
\end{aligned} \tag{3}$$

On cancelling the factors of 2, we get

$$\cos{(A + IB)} = \cos A \, \cosh B - I \sinh B \, \sin A \, . \tag{4}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.