Problem 6.6 on Page 93

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1 The Problem

On page 93 of NFCM [1], we find problem (6.6): Find the point of intersection of the line defined by the set of all $\mathbf{x}$ satisfying the equation

$$(\mathbf{x}-\mathbf{a})\wedge\mathbf{u}=0\,, \tag{1}$$

and the plane defined by the set of all points $\mathbf{y}$ defined by

$$(\mathbf{y}-\mathbf{b})\wedge\mathbf{B}=0\,, \tag{2}$$

where $\mathbf{u}\wedge\mathbf{B}\neq 0$.

2 Solution

We begin by assuming that there is a unique intersection point $\mathbf{p}$. Then, from (1)

$$(\mathbf{p}-\mathbf{a})\wedge\mathbf{u}=0 \quad\Longrightarrow\quad (\mathbf{p}-\mathbf{a})\cdot\mathbf{u}=(\mathbf{p}-\mathbf{a})\mathbf{u}=\mathbf{u}(\mathbf{p}-\mathbf{a})\,. \tag{3}$$

and from (2)

$$(\mathbf{p}-\mathbf{b})\wedge\mathbf{B}=0 \quad\Longrightarrow\quad \mathbf{p}\wedge\mathbf{B}=\mathbf{b}\wedge\mathbf{B}\,. \tag{4}$$

Virtually emplacing the vector $\mathbf{a}$ into this last equation gives

$$\mathbf{p}\wedge\mathbf{B}=[(\mathbf{p}-\mathbf{a})+\mathbf{a}]\wedge\mathbf{B}=(\mathbf{p}-\mathbf{a})\wedge\mathbf{B}+\mathbf{a}\wedge\mathbf{B}=\mathbf{b}\wedge\mathbf{B}\,. \tag{5}$$

Hence,

$$(\mathbf{p}-\mathbf{a})\wedge\mathbf{B}=(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}\,. \tag{6}$$

Now, dotting through on the left by $\mathbf{u}$, gives us

$$\mathbf{u}\cdot(\mathbf{p}-\mathbf{a})\wedge\mathbf{B}=\mathbf{u}(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}=(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}\,\mathbf{u}\,, \tag{7}$$

where $\mathbf{u}$ commutes with the pseudoscalar $(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}$. Expanding on the left gives us

$$\mathbf{u}\cdot(\mathbf{p}-\mathbf{a})\mathbf{B}-(\mathbf{p}-\mathbf{a})\mathbf{u}\cdot\mathbf{B}=(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}\,\mathbf{u}\,. \tag{8}$$

But from (3), we get that

$$(\mathbf{p}-\mathbf{a})\mathbf{u}\mathbf{B}-(\mathbf{p}-\mathbf{a})\mathbf{u}\cdot\mathbf{B}=(\mathbf{p}-\mathbf{a})\mathbf{u}\wedge\mathbf{B}=(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}\,\mathbf{u}\,. \tag{9}$$

But since $\mathbf{u}\wedge\mathbf{B}$ is a pseudoscalar, we can just divide through by it to get

$$\mathbf{p}=\mathbf{a}+\frac{(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}}{\mathbf{u}\wedge\mathbf{B}}\,\mathbf{u}\,, \tag{10}$$

and we are finished.

3 Conclusion

It amazes me that it is possible to solve explicitly for a point that is only defined implicitly, without using linear algebra.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.