Problem 8.2b on Page 117

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May 22, 2021

1 The Problem

On page 117 of NFCM [1], we find problem (8.2b): Show that

$$\mathbf{a}\cdot\nabla\hat{\mathbf{r}}=\frac{\hat{\mathbf{r}}\hat{\mathbf{r}}\wedge\mathbf{a}}{r}\,, \tag{1}$$

where

$$\mathbf{r}=\mathbf{x}-\mathbf{x}'\quad\text{and}\quad r=|\,\mathbf{x}-\mathbf{x}'\,|\,. \tag{2}$$

2 Lemma

$$\mathbf{a}\cdot\nabla\,\mathbf{x}=\mathbf{a}\,. \tag{3}$$

Proof: It's not often in geometric algebra to find a need to introduce a basis for a proof, but this time it will be useful. Let $\mathbf{x}=x_i\sigma_i$ (sum on the index i). Then $\mathbf{a}\cdot\nabla=a_j\partial_j$ (sum on the index j). So,

$$\mathbf{a}\cdot\nabla\,\mathbf{x}=a_j\partial_j x_i\sigma_i=a_j\delta_{ij}\sigma_i=a_j\sigma_j=\mathbf{a}\,. \tag{4}$$

Corollary: With $\mathbf{x}'$ being independent of $\mathbf{x}$ then

$$\mathbf{a}\cdot\nabla\,\mathbf{r}=\mathbf{a}\cdot\nabla\,(\mathbf{x}-\mathbf{x}')=\mathbf{a}\cdot\nabla\,\mathbf{x}=\mathbf{a}\,. \tag{5}$$

3 Solution

We start with

$$\begin{aligned}\mathbf{a}\cdot\nabla\,\hat{\mathbf{r}}&=\mathbf{a}\cdot\nabla\,r^{-1}\mathbf{r}\\&=(\mathbf{a}\cdot\nabla\,r^{-1})\mathbf{r}+r^{-1}\mathbf{a}\cdot\nabla\,\mathbf{r}\\&=\frac{-\mathbf{a}\cdot\hat{\mathbf{r}}}{r^2}\mathbf{r}+r^{-1}\mathbf{a}\\&=\frac{\mathbf{a}-\mathbf{a}\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}}}{r}\\&=\frac{\hat{\mathbf{r}}\,\hat{\mathbf{r}}\wedge\mathbf{a}}{r}\end{aligned} \tag{6}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.