

Problem 8.2c on Page 117

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1 The Problem

On page 117 of NFCM [1], we find problem (8.2c): Show that

$$\mathbf{a} \cdot \nabla (\hat{\mathbf{r}} \cdot \mathbf{a}) = \frac{|\hat{\mathbf{r}} \wedge \mathbf{a}|^2}{r}, \quad (1)$$

where

$$\mathbf{r} = \mathbf{x} - \mathbf{x}' \quad \text{and} \quad r = |\mathbf{x} - \mathbf{x}'|. \quad (2)$$

2 Lemmas (previously proved results)

$$\mathbf{a} \cdot \nabla \mathbf{r} = \mathbf{a}, \quad (3a)$$

$$\mathbf{a} \cdot \nabla \hat{\mathbf{r}} = \frac{\hat{\mathbf{r}} \hat{\mathbf{r}} \wedge \mathbf{a}}{r}. \quad (3b)$$

3 Solution

We start with

$$\begin{aligned} \mathbf{a} \cdot \nabla (\hat{\mathbf{r}} \cdot \mathbf{a}) &= \frac{1}{2} \mathbf{a} \cdot \nabla (\hat{\mathbf{r}} \mathbf{a} + \mathbf{a} \hat{\mathbf{r}}) \\ &= \frac{1}{2} \left[\frac{\hat{\mathbf{r}} \hat{\mathbf{r}} \wedge \mathbf{a} \mathbf{a}}{r} + \frac{\mathbf{a} \hat{\mathbf{r}} \hat{\mathbf{r}} \wedge \mathbf{a}}{r} \right] \\ &= \frac{1}{2} \left[\frac{-\hat{\mathbf{r}} \mathbf{a} \hat{\mathbf{r}} \wedge \mathbf{a}}{r} + \frac{\mathbf{a} \hat{\mathbf{r}} \hat{\mathbf{r}} \wedge \mathbf{a}}{r} \right] \\ &= \frac{1}{2} \frac{(\mathbf{a} \hat{\mathbf{r}} - \hat{\mathbf{r}} \mathbf{a}) \hat{\mathbf{r}} \wedge \mathbf{a}}{r} \\ &= \frac{(\mathbf{a} \wedge \hat{\mathbf{r}}) \hat{\mathbf{r}} \wedge \mathbf{a}}{r} \\ &= \frac{|\hat{\mathbf{r}} \wedge \mathbf{a}|^2}{r}. \end{aligned} \quad (4)$$

References

- [1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.