Problem 8.2f on Page 118

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1 The Problem

On page 118 of NFCM [1], we find problem (8.2f): Show that

$$\mathbf{a}\cdot\nabla\frac{1}{\mathbf{r}}=-\frac{1}{\mathbf{r}}\,\mathbf{a}\,\frac{1}{\mathbf{r}}\,, \tag{1}$$

where

$$\mathbf{r}=\mathbf{x}-\mathbf{x}' \quad \text{and} \quad r=|\,\mathbf{x}-\mathbf{x}'\,|\,. \tag{2}$$

2 Lemmas (previously proved results)

$$\mathbf{a}\cdot\nabla r=\mathbf{a}\cdot\hat{\mathbf{r}}\,, \tag{3a}$$

$$\mathbf{a}\cdot\nabla\hat{\mathbf{r}}=\frac{\hat{\mathbf{r}}\,\hat{\mathbf{r}}\wedge\mathbf{a}}{r}\,. \tag{3b}$$

3 Solution

We start with the useful transformation:

$$-\frac{1}{\mathbf{r}}\,\mathbf{a}\,\frac{1}{\mathbf{r}}=-\frac{1}{r^2}\hat{\mathbf{r}}\,\mathbf{a}\,\hat{\mathbf{r}}=-\frac{1}{r^2}(\hat{\mathbf{r}}\cdot\mathbf{a}\,\hat{\mathbf{r}}+\hat{\mathbf{r}}\wedge\mathbf{a}\,\hat{\mathbf{r}})\,. \tag{4}$$

Now,

$$\begin{aligned}\mathbf{a}\cdot\nabla\frac{1}{\mathbf{r}}&=\mathbf{a}\cdot\nabla\,(r^{-1}\hat{\mathbf{r}})\\&=-\frac{\hat{\mathbf{r}}\,\mathbf{a}\cdot\hat{\mathbf{r}}}{r^2}+\frac{1}{r^2}\hat{\mathbf{r}}\,\hat{\mathbf{r}}\wedge\mathbf{a}\\&=-\frac{\hat{\mathbf{r}}\,\mathbf{a}\cdot\hat{\mathbf{r}}+\hat{\mathbf{r}}\wedge\mathbf{a}\,\hat{\mathbf{r}}}{r^2}\,.\end{aligned} \tag{5}$$

On equating this last result with that of (4), we establish (1).

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.