

Problem 8.2h on Page 118

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1 The Problem

On page 118 of NFCM [1], we find problem (8.2h): Show that

$$\frac{1}{2}(\mathbf{a} \cdot \nabla)^2 \frac{1}{r^2} = \frac{3(\mathbf{a} \cdot \hat{\mathbf{r}})^2 - |\hat{\mathbf{r}} \wedge \mathbf{a}|^2}{r^4}, \quad (1)$$

where

$$\mathbf{r} = \mathbf{x} - \mathbf{x}' \quad \text{and} \quad r = |\mathbf{x} - \mathbf{x}'|. \quad (2)$$

2 Lemmas (previously proved results)

$$\mathbf{a} \cdot \nabla r = \mathbf{a} \cdot \hat{\mathbf{r}}, \quad (3a)$$

$$\mathbf{a} \cdot \nabla \frac{1}{r^2} = -2 \frac{\mathbf{a} \cdot \hat{\mathbf{r}}}{r^3}, \quad (3b)$$

$$\mathbf{a} \cdot \nabla (\hat{\mathbf{r}} \cdot \mathbf{a}) = \frac{|\hat{\mathbf{r}} \wedge \mathbf{a}|^2}{r}. \quad (3c)$$

3 Solution

Now,

$$\begin{aligned} \frac{1}{2}(\mathbf{a} \cdot \nabla)^2 \frac{1}{r^2} &= \frac{1}{2}(\mathbf{a} \cdot \nabla) \left(-2 \frac{\mathbf{a} \cdot \hat{\mathbf{r}}}{r^3} \right) \\ &= -(\mathbf{a} \cdot \nabla \frac{1}{r^3}) \mathbf{a} \cdot \hat{\mathbf{r}} - \frac{1}{r^3} (\mathbf{a} \cdot \nabla \mathbf{a} \cdot \hat{\mathbf{r}}) \\ &= \frac{3(\mathbf{a} \cdot \hat{\mathbf{r}})^2}{r^4} - \frac{1}{r^3} \frac{|\hat{\mathbf{r}} \wedge \mathbf{a}|^2}{r} \\ &= \frac{3(\mathbf{a} \cdot \hat{\mathbf{r}})^2 - |\hat{\mathbf{r}} \wedge \mathbf{a}|^2}{r^4}. \end{aligned} \quad (4)$$

References

- [1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.