Problem 8.2i on Page 118

P. Reany

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1 The Problem

On page 118 of NFCM [1], we find problem (8.2i): Show that

$$\tfrac{1}{6}(\mathbf{a}\cdot\nabla)^3\,\frac{1}{r^2} = -\frac{4(\mathbf{a}\cdot\hat{\mathbf{r}})^3 + 4|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2\mathbf{a}\cdot\hat{\mathbf{r}}}{r^5}\,, \tag{1}$$

where

$$\mathbf{r} = \mathbf{x} - \mathbf{x}' \quad \text{and} \quad r = |\,\mathbf{x}-\mathbf{x}'\,|\,. \tag{2}$$

2 Lemmas (previously proved results)

$$\mathbf{a}\cdot\nabla\,(\hat{\mathbf{r}}\cdot\mathbf{a}) = \frac{|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2}{r}\,, \tag{3a}$$

$$\mathbf{a}\cdot\nabla\,|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,| = -\frac{\hat{\mathbf{r}}\cdot\mathbf{a}\,|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|}{r}\,, \tag{3b}$$

$$\tfrac{1}{2}(\mathbf{a}\cdot\nabla)^2\,\frac{1}{r^2} = \frac{3(\mathbf{a}\cdot\hat{\mathbf{r}})^2 - |\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2}{r^4}\,. \tag{3c}$$

3 Solution

Now,

$$\begin{aligned}
\tfrac{1}{6}(\mathbf{a}\cdot\nabla)^3\,\frac{1}{r^2} &= \tfrac{1}{3}(\mathbf{a}\cdot\nabla)\big[\tfrac{1}{2}(\mathbf{a}\cdot\nabla)^2\,\frac{1}{r^2}\big] \\
&= \tfrac{1}{3}(\mathbf{a}\cdot\nabla)\big[\frac{3(\mathbf{a}\cdot\hat{\mathbf{r}})^2 - |\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2}{r^4}\big] \\
&= \tfrac{1}{3}\big(\frac{-4\mathbf{a}\cdot\hat{\mathbf{r}}}{r^5}\big)\big[3(\mathbf{a}\cdot\hat{\mathbf{r}})^2 - |\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2\big] + \frac{1}{3r^4}\Big[6(\mathbf{a}\cdot\hat{\mathbf{r}})\frac{|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2}{r} - 2|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|\big(\frac{-\mathbf{a}\cdot\hat{\mathbf{r}}|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|}{r}\big)\Big] \\
&= \frac{1}{3r^5}\big[-12(\mathbf{a}\cdot\hat{\mathbf{r}})^3 + 4\mathbf{a}\cdot\hat{\mathbf{r}}|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2 + 6(\mathbf{a}\cdot\hat{\mathbf{r}})|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2 + 2\mathbf{a}\cdot\hat{\mathbf{r}}|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2\big] \\
&= \frac{-4(\mathbf{a}\cdot\hat{\mathbf{r}})^3 + 4\mathbf{a}\cdot\hat{\mathbf{r}}|\,\hat{\mathbf{r}}\wedge\mathbf{a}\,|^2}{r^5}\,.
\end{aligned} \tag{4}$$

So, my solution is slightly different than the textbook solution.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.