Problem 8.4 on Page 118, Legendre Polynomials

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1 The Problem

On page 118 of NFCM [1], we find problem (8.4):
The Legendre Polynomials $P_n(\mathbf{xa})$ can be defined as the coefficients in the power series expansion

$$\begin{aligned}\frac{1}{|\,\mathbf{x}-\mathbf{a}\,|} &= \sum_{n=0}^{\infty}\frac{P_n(\hat{\mathbf{x}}\mathbf{a})}{|\,\mathbf{x}\,|^{n+1}} = \sum_{n=0}^{\infty}\frac{P_n(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+1}}\\ &= \frac{P_0(\mathbf{xa})}{|\,\mathbf{x}\,|} + \frac{P_1(\mathbf{xa})}{|\,\mathbf{x}\,|^3} + \frac{P_2(\mathbf{xa})}{|\,\mathbf{x}\,|^5} + \frac{P_3(\mathbf{xa})}{|\,\mathbf{x}\,|^7} + \cdots\end{aligned}$$

The series converges for $|\,\mathbf{x}\,| < |\,\mathbf{a}\,|$.

Use a Taylor expansion to evaluate the polynomials of lowest order, namely:

$$\begin{aligned}P_0(\mathbf{xa}) &= 1\,,\\ P_1(\mathbf{xa}) &= \mathbf{a}\cdot\mathbf{x}\,,\\ P_2(\mathbf{xa}) &= (\mathbf{a}\cdot\mathbf{x})^2 - \tfrac{1}{2}|\,\mathbf{x}\wedge\mathbf{a}\,|^2\,,\\ P_3(\mathbf{xa}) &= (\mathbf{a}\cdot\mathbf{x})^3 - \tfrac{3}{2}(\mathbf{a}\cdot\mathbf{x})|\,\mathbf{x}\wedge\mathbf{a}\,|^2\,.\end{aligned}$$

The $P_n(\mathbf{xa})$ are polynomials of vectors. Show that they are homogeneous functions of degree n in the variable $|\,\mathbf{x}\,|$, that is,

$$P_n(\mathbf{xa}) = |\,\mathbf{x}\,|^n P_n(\hat{\mathbf{x}}\mathbf{a}) = |\,\mathbf{x}\,|^n|\,\mathbf{a}\,|^n P_n(\hat{\mathbf{x}}\hat{\mathbf{a}})\,. \tag{1}$$

2 Lemmas (previously proved results)

$$\mathbf{a}\cdot\nabla\,|\,\mathbf{x}\,| = \mathbf{a}\cdot\hat{\mathbf{x}}\,, \tag{2a}$$

$$\mathbf{a}\cdot\nabla\,\frac{1}{|\,\mathbf{x}\,|} = -\frac{\mathbf{a}\cdot\hat{\mathbf{x}}}{|\,\mathbf{x}\,|^2}\,, \tag{2b}$$

$$\mathbf{a}\cdot\nabla\,(\mathbf{a}\cdot\hat{\mathbf{x}}) = \frac{|\,\hat{\mathbf{x}}\wedge\mathbf{a}\,|^2}{|\,\mathbf{x}\,|}\,, \tag{2c}$$

$$\mathbf{a}\cdot\nabla\,|\,\hat{\mathbf{x}}\wedge\mathbf{a}\,| = -\frac{\mathbf{a}\cdot\hat{\mathbf{x}}\,|\,\hat{\mathbf{x}}\wedge\mathbf{a}\,|}{|\,\mathbf{x}\,|}\,, \tag{2d}$$

$$\tfrac{1}{2}(\mathbf{a}\cdot\nabla)^2\,\frac{1}{|\,\mathbf{x}\,|^2} = \frac{3(\mathbf{a}\cdot\hat{\mathbf{x}})^2 - |\,\hat{\mathbf{x}}\wedge\mathbf{a}\,|^2}{|\,\mathbf{x}\,|^4}\,. \tag{2e}$$

3 Solution to the first part

We begin by expanding $\frac{1}{|\,\mathbf{x}-\mathbf{a}\,|}$ in a Taylor series

$$\frac{1}{|\,\mathbf{x}-\mathbf{a}\,|} = \sum_{n=0}^{\infty} \frac{(-\mathbf{a}\cdot\nabla)^n}{n!}\frac{1}{|\,\mathbf{x}\,|} = \sum_{n=0}^{\infty} \frac{P_n(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+1}} \tag{3}$$

Setting $n=0$, we get

$$\frac{1}{|\,\mathbf{x}\,|} = \frac{P_0(\mathbf{xa})}{|\,\mathbf{x}\,|}\,. \tag{4}$$

Solving for $P_0(\mathbf{xa})$ we get

$$P_0(\mathbf{xa}) = 1\,. \quad \checkmark \tag{5}$$

Setting $n=1$, we get

$$-\mathbf{a}\cdot\nabla\frac{1}{|\,\mathbf{x}\,|} = \frac{-\mathbf{a}\cdot\nabla|\,\mathbf{x}\,|}{|\,\mathbf{x}\,|^2} = \frac{\mathbf{a}\cdot\hat{\mathbf{x}}}{|\,\mathbf{x}\,|^2} = \frac{\mathbf{a}\cdot\mathbf{x}}{|\,\mathbf{x}\,|^3}\,. \tag{6}$$

Solving for $P_1(\mathbf{xa})$ we get

$$P_1(\mathbf{xa}) = \mathbf{a}\cdot\mathbf{x}\,. \quad \checkmark \tag{7}$$

Setting $n=2$, we get (using the previous result, Eq. (6))

$$\begin{aligned}
\tfrac{1}{2}(-\mathbf{a}\cdot\nabla)^2\frac{1}{|\,\mathbf{x}\,|} &= (-\mathbf{a}\cdot\nabla)\tfrac{1}{2}(-\mathbf{a}\cdot\nabla)\frac{1}{|\,\mathbf{x}\,|} \\
&= \tfrac{1}{2}(-\mathbf{a}\cdot\nabla)\frac{\mathbf{a}\cdot\hat{\mathbf{x}}}{|\,\mathbf{x}\,|^2} = \cdots \\
&= \frac{(\mathbf{a}\cdot\hat{\mathbf{x}})^2 - \frac{1}{2}|\,\hat{\mathbf{x}}\wedge\mathbf{a}\,|^2}{|\,\mathbf{x}\,|^3} && (8)\\
&= \frac{(\mathbf{a}\cdot\mathbf{x})^2 - \frac{1}{2}|\,\mathbf{x}\wedge\mathbf{a}\,|^2}{|\,\mathbf{x}\,|^5}\,. && (9)
\end{aligned}$$

Solving for $P_2(\mathbf{xa})$ we get

$$P_2(\mathbf{xa}) = (\mathbf{a}\cdot\mathbf{x})^2 - \tfrac{1}{2}|\,\mathbf{x}\wedge\mathbf{a}\,|^2\,. \quad \checkmark \tag{10}$$

Setting $n=3$, we get (using the previous result, Eq. (8))

$$\begin{aligned}
\tfrac{1}{6}(-\mathbf{a}\cdot\nabla)^2\frac{1}{|\,\mathbf{x}\,|} &= \tfrac{1}{3}(-\mathbf{a}\cdot\nabla)\tfrac{1}{2}(-\mathbf{a}\cdot\nabla)^2\frac{1}{|\,\mathbf{x}\,|} \\
&= \tfrac{1}{3}(-\mathbf{a}\cdot\nabla)\Big[\frac{(\mathbf{a}\cdot\hat{\mathbf{x}})^2 - \frac{1}{2}|\,\hat{\mathbf{x}}\wedge\mathbf{a}\,|^2}{|\,\mathbf{x}\,|^3}\Big] = \cdots \\
&= \frac{1}{|\,\mathbf{x}\,|^4}\big[(\mathbf{a}\cdot\hat{\mathbf{x}})^3 - \tfrac{3}{2}(\mathbf{a}\cdot\hat{\mathbf{x}})|\,\hat{\mathbf{x}}\wedge\mathbf{a}\,|^2\big] && (11)\\
&= \frac{(\mathbf{a}\cdot\mathbf{x})^3 - \frac{3}{2}(\mathbf{a}\cdot\mathbf{x})|\,\mathbf{x}\wedge\mathbf{a}\,|^2}{|\,\mathbf{x}\,|^7}\,. && (12)
\end{aligned}$$

Solving for $P_3(\mathbf{xa})$ we get

$$P_3(\mathbf{xa}) = (\mathbf{a}\cdot\mathbf{x})^3 - \tfrac{3}{2}(\mathbf{a}\cdot\mathbf{x})|\,\mathbf{x}\wedge\mathbf{a}\,|^2\,. \quad \checkmark \tag{13}$$

4 Proof to homogenous part

I will humbly attempt a solution to the question involving the homogeneity of $P_n(\mathbf{xa})$ in the variable $|\,\mathbf{x}\,|$ only. My proof will be by induction on the degree of $P_n(\mathbf{xa})$. The main idea of my proof is to use the calculated expression for $P_n(\mathbf{xa})$ to be the kernel of the calculation for $P_{n+1}(\mathbf{xa})$ as well. In fact, we used this technique above to calculate progressive values for $P_n(\mathbf{xa})$.

Proof by induction: We begin by assuming that $P_n(\mathbf{xa})$ is a homogeneous polynomial in the variable $|\,\mathbf{x}\,|$ of degree n, and that

$$P_n'(\mathbf{xa}) \equiv (-\mathbf{a}\cdot\nabla)P_n(\mathbf{xa}) \tag{14}$$

is a homogeneous polynomial in the variable $|\,\mathbf{x}\,|$ of degree $n-1$. So, how did we move up the polynomial chain? We first state the basics:

$$\frac{P_n(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+1}} \equiv \frac{1}{n!}(-\mathbf{a}\cdot\nabla)^n\frac{1}{|\,\mathbf{x}\,|} \tag{15}$$

Remember that the above equation defines the polynomials $P_n(\mathbf{xa})$. Now, what happens when we let $n \to n+1$? We get

$$\frac{P_{n+1}(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+3}} \equiv \frac{1}{(n+1)!}(-\mathbf{a}\cdot\nabla)^{n+1}\frac{1}{|\,\mathbf{x}\,|} \tag{16}$$

By performing a couple identity operations, we get

$$\begin{aligned}
\frac{P_{n+1}(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+3}} &= \frac{n!}{(n+1)!}(-\mathbf{a}\cdot\nabla)\left[\frac{1}{n!}(-\mathbf{a}\cdot\nabla)^n\frac{1}{|\,\mathbf{x}\,|}\right] \\
&= \frac{1}{n+1}(-\mathbf{a}\cdot\nabla)\left[\frac{P_n(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+1}}\right] \quad \text{(from (15))} \\
&= \frac{1}{n+1}\left[\frac{P_n'(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+1}} - (2n+1)\frac{P_n(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+2}}(-\mathbf{a}\cdot\nabla)|\,\mathbf{x}\,|\right] \\
&= \frac{1}{n+1}\left[\frac{P_n'(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+1}} + (2n+1)\frac{P_n(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+2}}(\mathbf{a}\cdot\hat{\mathbf{x}})\right].
\end{aligned} \tag{17}$$

From this we get

$$\begin{aligned}
P_{n+1}(\mathbf{xa}) &= |\,\mathbf{x}\,|^{2n+3}\left\{\frac{1}{n+1}\frac{P_n'(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+1}} + \frac{2n+1}{n+1}\frac{P_n(\mathbf{xa})}{|\,\mathbf{x}\,|^{2n+2}}(\mathbf{a}\cdot\hat{\mathbf{x}})\right\} \\
&= \frac{1}{n+1}P_n'(\mathbf{xa})|\,\mathbf{x}\,|^2 + \frac{2n+1}{n+1}(\mathbf{a}\cdot\hat{\mathbf{x}})P_n(\mathbf{xa})|\,\mathbf{x}\,|.
\end{aligned} \tag{18}$$

Now, since $P_n'(\mathbf{xa})$ is assumed to be of degree $n-1$ in $|\,\mathbf{x}\,|$, then $P_n'(\mathbf{xa})|\,\mathbf{x}\,|^2$ is of degree $n+1$, and so the first term on the RHS is of degree $n+1$. Furthermore, since $\mathbf{a}\cdot\hat{\mathbf{x}}$ is of degree zero in $|\,\mathbf{x}\,|$, then the second term is of degree $n+1$, as well. Hence, $P_{n+1}(\mathbf{xa})$ is of degree $n+1$ in $|\,\mathbf{x}\,|$.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.