Sum of Angles Trig Formulas

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1 The Problem

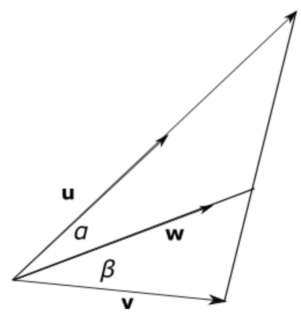

Figure 1. Vectors **u**, **v**, **w** are all unit vectors.

Our problem here is to use geometric algebra to find the cosine and sine of a sum of angles, as provided in the Figure, angles α and β. The vectors **u**, **v**, **w** are all unit vectors, meaning that

$$\mathbf{u}^2 = \mathbf{v}^2 = \mathbf{w^2} = \mathbf{1}\,. \tag{1}$$

2 The Solution

The best way to utilize geometric algebra is through its associative geometric product.[1]

$$\mathbf{uv} = \mathbf{uw^2v} = (\mathbf{uw})(\mathbf{wv})\,. \tag{2}$$

Expanding these products, we get

$$\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v} = (\mathbf{u}\cdot\mathbf{w} + \mathbf{u}\wedge\mathbf{w})(\mathbf{w}\cdot\mathbf{v} + \mathbf{w}\wedge\mathbf{v})\,. \tag{3}$$

Equating the scalar parts gives

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}\mathbf{w}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{w}\cdot\mathbf{w}\wedge\mathbf{v}\,. \tag{4}$$

Equating the bivector parts gives

$$\mathbf{u}\wedge\mathbf{v} = \mathbf{u}\cdot\mathbf{w}\,\mathbf{w}\wedge\mathbf{v} + \mathbf{w}\cdot\mathbf{v}\,\mathbf{u}\wedge\mathbf{w}\,. \tag{5}$$

On replacing the dot products with cosines and the wedge products with sines, for Eq. (4), we get

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\,, \tag{6}$$

[1]Probably the easiest way to derive these formulas without using geometric algebra is by using complex numbers and the Euler Formula $e^{i\theta} = \cos\theta + i\sin\theta$ together with $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$.

where the minus sign came from the fact that the bivectors have the same orientation, that is, it's equivalent to $\mathbf{i} \cdot \mathbf{i} = -1$.[2] And for Eq. (5), we get

$$\sin(\alpha + \beta) = \cos\beta \sin\alpha + \cos\alpha \sin\beta\,. \tag{7}$$

And this ends the direct use of geometric algebra.

Nevertheless, we should add in to our presentation the formula for the tangent of $\alpha + \beta$, which can be done without further appeal to geometric algebra:

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\ &= \frac{\cos\beta \sin\alpha + \cos\alpha \sin\beta}{\cos\alpha \cos\beta - \sin\alpha \sin\beta}\,. \end{aligned} \tag{8}$$

On dividing the numerator and denominator by $\cos\alpha \cos\beta$ and simplifying, we get

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}\,. \tag{9}$$

References

[1] D. Hestenes, *New Foundations in Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.

[2] Or, put more accurately in our particular case, that $(-\mathbf{i}) \cdot (-\mathbf{i}) = -1$, as both bivectors have negative orientations.