Problem 2.5 on Page 134

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May 26, 2021

1 The Problem

On page 134 of NFCM [1], we find Problem (2.5): Find the minimum initial speed v_0 needed for a projectile to reach a target with horizontal range x and elevation y. Determine also the firing angle θ, the time of flight t, and the final speed v of the projectile. Specifically, show that

$$v = [\, g(r - y)\,]^{1/2}\,, \qquad v_0 = [\, g(r + y)\,]^{1/2}\,, \qquad \tan\,\theta = \left[\frac{r + y}{r - y}\right]^{1/2}. \tag{1}$$

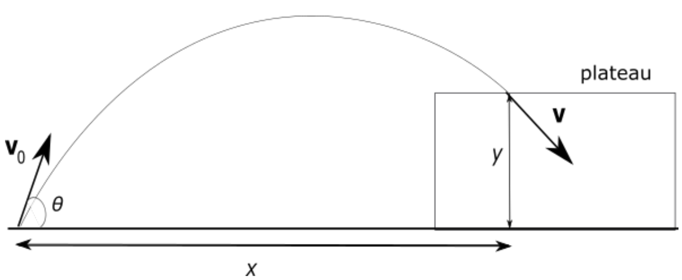

Figure 1. We've looking for the solution of minimum v_0 to attain its range values of x in the horizontal direction and y in the vertical direction.

2 Solution

Since

$$2\mathbf{g} \cdot \mathbf{r} = v^2 - v_0^2\,, \tag{2}$$

and

$$y = -\hat{\mathbf{g}} \cdot \mathbf{r}\,, \tag{3}$$

then

$$y = \frac{v_0^2 - v^2}{2g}\,, \tag{4}$$

and

$$v_0^2 = 2gy - v^2\,. \tag{5}$$

Now, employing Eq. (2.12) on page 129, we find that

$$\begin{aligned} r_{\max} &= \frac{v_0^2}{g} \frac{1}{1 - \hat{\mathbf{g}} \cdot \hat{\mathbf{r}}} \\ &= \frac{v_0^2}{g} \frac{1}{1 + yg} \end{aligned} \tag{6}$$

Hence,

$$v_0 = [\, g(r+y)\,]^{1/2} \,. \tag{7}$$

And on solving for v from (5), we get

$$\begin{aligned} v^2 &= v_0^2 - 2gy \\ &= g(r+y) - 2gy \\ &= g(r-y)\,, \end{aligned} \tag{8}$$

and therefore,

$$v = [\, g(r-y)\,]^{1/2} \,. \tag{9}$$

The minimum speed necessary to attain the given x and y guarantees a speed of maximum range, and hence that $\mathbf{v}_0$ and $\mathbf{v}$ are orthogonal to each other. Therefore, as in Figure 2, θ and $\overline{\theta}$ are complementary angles.

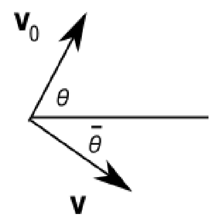

Figure 2. The angles θ and $\overline{\theta}$ are complementary.

From Equation (2.15), we have that

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{g}\, t \,. \tag{10}$$

Dotting this by $\hat{\mathbf{x}}$, we get that

$$v \sin\theta = v_0 \cos\,\theta \,. \tag{11}$$

From this we get

$$\tan\,\theta = \left[\frac{r+y}{r-y}\right]^{1/2} . \tag{12}$$

Now to calculate the flight time t. Let's try to find it with Eq. (2.15) from the text:

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{g}\, t \,. \tag{13}$$

From this we get

$$g\, t = \hat{\mathbf{g}} \cdot \mathbf{v} - \hat{\mathbf{g}} \cdot \mathbf{v}_0 \,. \tag{14}$$

Keepng in mind that $\hat{\mathbf{g}}$ points in the $-y$ direction, we get

$$\begin{aligned} g\, t &= -v \cos\,\theta + v_0 \sin\,\theta \\ &= -[\, g(r-y)\,]^{1/2} \cos\,\theta + [\, g(r+y)\,]^{1/2} \sin\,\theta \,, \end{aligned} \tag{15}$$

where we used (7) and (9). Finally, we can calculate $\cos\,\theta$ and $\sin\,\theta$ from (12), yielding

$$\cos\,\theta = \Big[\,\frac{r-y}{2r}\,\Big]^{1/2}, \qquad \sin\,\theta = \Big[\,\frac{r+y}{2r}\,\Big]^{1/2} . \tag{16}$$

Substituting these into (15), we get

$$t = \Big[\,\frac{2}{gr}\,\Big]^{1/2} y \,. \tag{17}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.