Problem 3.1 on Page 138

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1 The Problem

On page 138 of NFCM [1], we find problem (3.1): Show that if $\mathbf{g} \cdot \mathbf{v}_0 < 0$, then a particle subject to Equation (3.1) reaches a maximum height in time

$$t_{\mathrm{m}} = \gamma^{-1} \log\left(1 - \mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1}\right), \tag{1}$$

where $\mathbf{v}_\infty \equiv \gamma^{-1}\mathbf{g}$, the terminal velocity. Show that the displacement to maximum $\mathbf{r}_{\mathrm{m}}$ is given by

$$\gamma \mathbf{r}_{\mathrm{m}} = \mathbf{v}_\infty \, \log\left(1 - \mathbf{v}_0 \cdot \mathbf{v}_\infty\right) - \frac{\mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1}(\mathbf{v}_0 - \mathbf{v}_\infty)}{1 - \mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1}}\,. \tag{2}$$

Lastly, defining x_{m} to be the horizontal coordinate of the maximum height point, and x_∞ to be the distance from the initial position to the vertical asymptote, show that

$$\frac{x_{\mathrm{m}}}{x_\infty} = \frac{-\mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1}}{1 - \mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1}} = \frac{-\mathbf{v}_0 \cdot \mathbf{g}^{-1}}{\gamma^{-1} - \mathbf{v}_0 \cdot \mathbf{g}^{-1}}\,. \tag{3}$$

2 The Solution

We begin with Eq. (3.2), which derives from Equation (3.1),

$$\mathbf{v}(t) = \mathbf{g}\frac{1 - e^{-\gamma t}}{\gamma} + \mathbf{v}_0\, e^{-\gamma t}\,. \tag{4}$$

Now, the condition for maximum height is $\mathbf{v} \cdot \hat{\mathbf{g}} = 0$. Therefore, on dotting this last equation by $\mathbf{g}^{-1}$, we get

$$\mathbf{v} \cdot \mathbf{g}^{-1} = \frac{1 - e^{-\gamma t}}{\gamma} + \mathbf{v}_0 \cdot \mathbf{g}^{-1}\, e^{-\gamma t} = 0\,, \tag{5}$$

where the time $t = t_{\mathrm{m}}$. A useful relation we derive from this is

$$e^{-\gamma t_{\mathrm{m}}} = (1 - \mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1})^{-1}\,. \tag{6}$$

And from this we can solve for t_{m}

$$t_{\mathrm{m}} = \gamma^{-1} \log\left(1 - \mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1}\right). \tag{7}$$

Lemma:

$$1 - (1 - \mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1})^{-1} = \frac{-\mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1}}{1 - \mathbf{v}_0 \cdot \mathbf{v}_\infty^{-1}}\,. \tag{8}$$

Now, to get the displacement to maximum $\mathbf{r}_{\rm m}$, we substitute $t_{\rm m}$ into Eq. (3.4) in the text, to get

$$\mathbf{r}_{\rm m} = \mathbf{g}\,\frac{(1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1})^{-1} + \log(1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1}) - 1}{\gamma^2} + \mathbf{v}_0\frac{1-(1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1})^{-1}}{\gamma}\,. \tag{9}$$

This can be rewritten as at first

$$\gamma\mathbf{r}_{\rm m} = \mathbf{v}_\infty\big[\,(1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1})^{-1} + \log(1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1}) - 1\,\big] + \mathbf{v}_0\big[\,1-(1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1})^{-1}\,\big]\,, \tag{10}$$

and then as

$$\gamma\mathbf{r}_{\rm m} = \mathbf{v}_\infty\big[\,(1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1})^{-1} + \log(1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1}) - 1\,\big] + \mathbf{v}_0\big[\,1-(1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1})^{-1}\,\big]\,, \tag{11}$$

and (employing the Lemma) finally as

$$\gamma\mathbf{r}_{\rm m} = \mathbf{v}_\infty\ \log(1-\mathbf{v}_0\cdot\mathbf{v}_\infty) - \frac{\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1}(\mathbf{v}_0-\mathbf{v}_\infty)}{1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1}}\,. \tag{12}$$

The x-coordinate of $\mathbf{r}_{\rm m}$ is given as $|\,\mathbf{r}_{\rm m}\wedge\hat{\mathbf{g}}\,|$, hence

$$\gamma\,x_{\rm m} = \gamma\,|\,\mathbf{r}_{\rm m}\wedge\hat{\mathbf{g}}\,| = \frac{|\,\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1}\,||\,\mathbf{v}_0\wedge\hat{\mathbf{g}}\,|}{1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1}}\,. \tag{13}$$

But

$$x_\infty = |\,\mathbf{r}_\infty\wedge\hat{\mathbf{g}}\,| = |\,\mathbf{v}_0\wedge\hat{\mathbf{g}}\,|/\gamma\,. \tag{14}$$

Hence, dividing (13) by $\gamma x_\infty = |\,\mathbf{v}_0\wedge\hat{\mathbf{g}}\,|$, we get

$$\frac{x_{\rm m}}{x_\infty} = \frac{-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1}}{1-\mathbf{v}_0\cdot\mathbf{v}_\infty^{-1}} = \frac{-\mathbf{v}_0\cdot\mathbf{g}^{-1}}{\gamma^{-1}-\mathbf{v}_0\cdot\mathbf{g}^{-1}}\,. \tag{15}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.