Problem (7.1) on Page 163

P. Reany

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1 The Problem

On page 163 of NFCM [1], we find Problem (7.1): Solve Eq. (7.3) for constant fields by the method of undetermined coefficients. Assume the solution is of the form:

$$\mathbf{v} = \mathbf{a}e^{\mathbf{\Omega}t} + \mathbf{b}t + \mathbf{c}\,, \tag{1}$$

with initial condition $\mathbf{v}(0) = \mathbf{v}_0$. Note: by balancing grades across (1), we require that

$$\mathbf{a}\wedge\mathbf{\Omega} = 0 \quad \text{hence} \quad \mathbf{a}\mathbf{\Omega} = \mathbf{a}\cdot\mathbf{\Omega}\,. \tag{2}$$

2 Solution

On applying this initial condition, we get that

$$\mathbf{v}_0 = \mathbf{a} + \mathbf{c}\,, \tag{3}$$

or

$$\mathbf{c} = \mathbf{v}_0 - \mathbf{a}\,. \tag{4}$$

Now, Eq. (7.3) is

$$\dot{\mathbf{v}} = \mathbf{g} + \mathbf{v}\cdot\Omega\,. \tag{5}$$

On substituting (1) into this last equation, we have

$$\mathrm{D}_t[\,\mathbf{a}e^{\mathbf{\Omega}t} + \mathbf{b}t + \mathbf{c}\,] = \mathbf{g} + [\,\mathbf{a}e^{\mathbf{\Omega}t} + \mathbf{b}t + \mathbf{c}\,]\cdot\Omega\,. \tag{6}$$

Expanding this, we get

$$\mathbf{a}\mathbf{\Omega}e^{\mathbf{\Omega}t} + \mathbf{b} = \mathbf{g} + (\mathbf{a}e^{\mathbf{\Omega}t})\cdot\Omega + \mathbf{b}\cdot\mathbf{\Omega}t + \mathbf{c}\cdot\mathbf{\Omega}\,. \tag{7}$$

On setting t to zero, we get

$$\mathbf{a}\mathbf{\Omega} + \mathbf{b} = \mathbf{g} + \mathbf{a}\cdot\mathbf{\Omega} + \mathbf{c}\cdot\mathbf{\Omega}\,. \tag{8}$$

This simplifies to

$$\mathbf{b} = \mathbf{g} + \mathbf{c}\cdot\mathbf{\Omega}\,. \tag{9}$$

By subtracting this last equation from (7), we get

$$\mathbf{a}\mathbf{\Omega}e^{\mathbf{\Omega}t} = (\mathbf{a}e^{\mathbf{\Omega}t})\cdot\mathbf{\Omega} + \mathbf{b}\cdot\mathbf{\Omega}\,t\,. \tag{10}$$

Since $\mathbf{a}e^{\mathbf{\Omega}t}$ is a vector in the $\mathbf{\Omega}$-plane, then $(\mathbf{a}e^{\mathbf{\Omega}t})\cdot\mathbf{\Omega} = \mathbf{a}\mathbf{\Omega}e^{\mathbf{\Omega}t}$, therefore, this last equation becomes

$$0 = \mathbf{b}\cdot\mathbf{\Omega}t\,, \tag{11}$$

from which we have that

$$\mathbf{b}\cdot\mathbf{\Omega} = 0\,. \tag{12}$$

Using (9), we get

$$(\mathbf{g}+\mathbf{c}\cdot\mathbf{\Omega})\cdot\mathbf{\Omega} = 0\,, \tag{13}$$

which expands to

$$[\,\mathbf{g}+(\mathbf{v}_0-\mathbf{a})\cdot\mathbf{\Omega}\,]\cdot\mathbf{\Omega} = 0\,. \tag{14}$$

Now we can drop the dot products by using only the components in the $\mathbf{\Omega}$-plane:

$$[\,\mathbf{g}_\perp+(\mathbf{v}_{0\perp}-\mathbf{a})\mathbf{\Omega}\,]\mathbf{\Omega} = 0\,. \tag{15}$$

It's clear from this that we must set the expression inside the square brackets to zero.

$$\mathbf{g}_\perp+(\mathbf{v}_{0\perp}-\mathbf{a})\mathbf{\Omega} = 0\,. \tag{16}$$

Solving this for $\mathbf{a}$, we get

$$\mathbf{a} = \mathbf{g}_\perp\mathbf{\Omega}^{-1}+\mathbf{v}_{0\perp} = \mathbf{g}\cdot\mathbf{\Omega}^{-1}+\mathbf{v}_{0\perp}\,. \tag{17}$$

We have only now to finish solving for $\mathbf{b}$ and $\mathbf{c}$.

Now,

$$\begin{aligned}\mathbf{b} &= \mathbf{g}+\mathbf{c}\cdot\mathbf{\Omega}\\ &= \mathbf{g}+(\mathbf{v}_{0\perp}-\mathbf{a})\mathbf{\Omega}\\ &= \mathbf{g}-\mathbf{g}_\perp\mathbf{\Omega}^{-1}\mathbf{\Omega} \quad \text{(from Eq. (17))}\\ &= \mathbf{g}-\mathbf{g}_\perp\\ &= \mathbf{g}_\parallel\,. \end{aligned} \tag{18}$$

And finally,

$$\begin{aligned}\mathbf{c} &= \mathbf{v}_0-(\mathbf{g}_\perp\mathbf{\Omega}^{-1}+\mathbf{v}_{0\perp})\\ &= \mathbf{v}_{0\parallel}-\mathbf{g}_\perp\mathbf{\Omega}^{-1}\\ &= \mathbf{v}_{0\parallel}-\mathbf{g}\cdot\mathbf{\Omega}^{-1}\,. \end{aligned} \tag{19}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.