Problem 1.1 on Page 198

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People often overlook the obvious.
— Doctor Who, "Pirate Planet"

1 The Problem

On page 198 of NFCM [1], we find Problem (1.1): Prove that if

$$\dot{\mathbf{L}} = 0\,, \tag{1}$$

then both the magnitude L and the direction $\hat{\mathbf{L}}$ of the angular momentum are constants of the motion.

2 Solution

Alternatively, we're being asked to show that, respectively, $\dot{L} = 0$ and $\dot{\hat{\mathbf{L}}} = 0$. But in addition to assuming (1), we'll assume that $L \neq 0$. Now, a good place to start is to acknowldge that

$$-L^2 = \mathbf{L}^2\,. \tag{2}$$

On differentiating this by time, we get

$$-2L\dot{L} = 2\mathbf{L} \cdot \dot{\mathbf{L}}\,. \tag{3}$$

From Eq. (1) we conclude that the RHS of this last equation is zero, hence, the LHS is also zero, by which we conclude that

$$\dot{L} = 0\,, \tag{4}$$

since $L \neq 0$. Thus, we have shown that the magnitude of the angular momentum is a constant of the motion.

Our final proof begins by differentiating the following by time:

$$\mathbf{L} = L\hat{\mathbf{L}}\,, \tag{5}$$

to get

$$\dot{\mathbf{L}} = \dot{L}\hat{\mathbf{L}} + L\dot{\hat{\mathbf{L}}} = 0\,, \tag{6}$$

where we used (1). Employing (4), we get $L\dot{\hat{\mathbf{L}}} = 0$, and since $L \neq 0$, we can conclude that

$$\dot{\hat{\mathbf{L}}} = 0\,, \tag{7}$$

which is what we were to show.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.