Problem 2.2 on Page 203

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June 7, 2021

1 The Problem

On page 203 of NFCM [1], we find Problem (2.2): To relate the constants in the two different Equations (2.2) and (2.7) [of the text] for an ellipse (See Figure 2-6.14a, page 95):

a) Evaluate r at the points $\mathbf{x} = \pm\mathbf{a}$ to show that

$$\ell = a(1 - \epsilon^2)\,; \tag{1}$$

b) and that $a\epsilon = |\,\mathbf{a}\,|\,|\,\boldsymbol{\epsilon}\,|$ is the distance from the center of the ellipse to the foci;

c) evaluate r at $\mathbf{x} = \mathbf{b}$ to show that

$$b^2 = (1 - \epsilon^2)a^2 = a\ell\,. \tag{2}$$

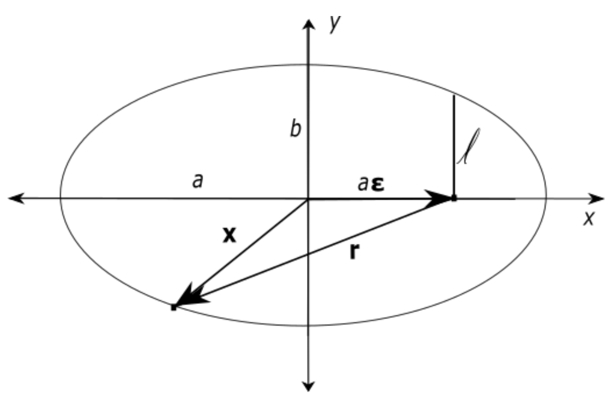

Figure 1. This is the text's Figure 2-6.14a, page 95.

The equations referenced above are, first, Eq. (2.2):

$$r = \frac{\ell}{1 + \boldsymbol{\epsilon}\cdot\hat{\mathbf{r}}}\,, \tag{3}$$

and, second, Eq. (2.7):

$$\mathbf{x} = \mathbf{a}\cos\phi + \mathbf{b}\sin\phi\,, \tag{4}$$

2 Solution

Part (a) Our plan is to set

$$2a = r_{\mathbf{a}} + r_{-\mathbf{a}}\,, \tag{5}$$

where $r_{\mathbf{a}}$ is the distance between the foci on the $+x$-axis (at point $a\boldsymbol{\epsilon}$) and the point $\mathbf{a}$ (where $\boldsymbol{\epsilon}\cdot\hat{\mathbf{r}} = \epsilon$), and where $r_{-\mathbf{a}}$ is the distance between the foci on the $+x$-axis and the point $-\mathbf{a}$ (where $\boldsymbol{\epsilon}\cdot\hat{\mathbf{r}} = -\epsilon$); hence, using (3), we get

$$2a = r_{\mathbf{a}} + r_{-\mathbf{a}} = \frac{\ell}{1+\epsilon} + \frac{\ell}{1-\epsilon} = \frac{2\ell}{1-\epsilon^2}\,. \tag{6}$$

Solving this for ℓ, we get

$$\ell = a(1-\epsilon^2)\,. \tag{7}$$

Part (b) Define D as the distance from the origin to the foci on the $+x$-axis, then

$$\begin{aligned} D &= a - r_{\mathbf{a}} \\ &= a - \frac{\ell}{1+\epsilon} \\ &= a - \frac{a(1-\epsilon^2)}{1+\epsilon} \quad \text{(from (7))} \\ &= a\epsilon\,. \end{aligned} \tag{8}$$

Part (c)

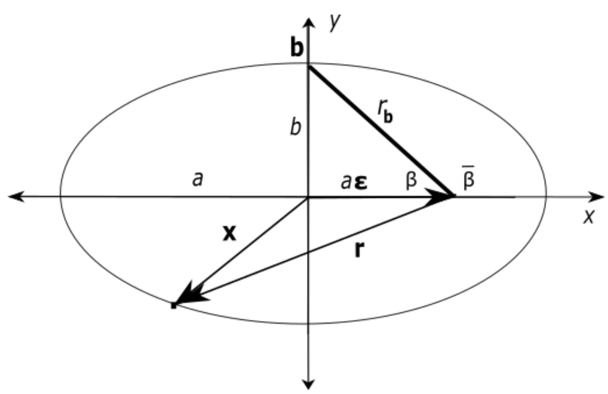

Figure 2. This is the adjusted figure to help solve for b. The angle $\overline{\beta}$ is the angle between $\boldsymbol{\epsilon}$ and $\hat{\mathbf{r}}$ when $\hat{\mathbf{r}}$ points to point $\mathbf{b}$; hence, $\cos\overline{\beta} = -\cos\beta$. Vectors $\mathbf{x}$ and $\mathbf{r}$ are left in their generic positions.

Define $r_{\mathbf{a}}$ as the distance from the foci to the point $\mathbf{b}$ on the $+y$-axis, then

$$\begin{aligned} r_{\mathbf{a}} &= \frac{\ell}{1+\boldsymbol{\epsilon}\cdot\hat{\mathbf{r}}} \\ &= \frac{\ell}{1+\epsilon\cos\overline{\beta}} \\ &= \frac{\ell}{1-\epsilon\cos\beta}\,. \end{aligned} \tag{9}$$

But from Figure 2, we calculate that

$$\cos\beta = \frac{a\epsilon}{r_{\mathbf{b}}}\,. \tag{10}$$

On substituting this into (9), and solving for $r_{\mathbf{a}}$, we get

$$r_{\mathbf{b}} = a\,. \tag{11}$$

We now have a relation obtained for b from the Pythagorean Theorem, namely

$$a^2 = b^2 + (a\epsilon)^2\,. \tag{12}$$

On solving this for b^2 and using (7), we have that

$$b^2 = a^2 - a^2\epsilon^2 = a^2(1-\epsilon^2) = a\ell\,. \tag{13}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.