Equation (3.20) on Page 210

P. Reany

June 27, 2021

1 Introduction

On page 210 of NFCM [1], we find the result (Eq. (3.20))

$$\mathbf{v}_f = \left(\frac{\mathbf{L}v_0 + k}{\mathbf{L}v_0 - k} \right) \mathbf{v}_0 \,, \tag{1}$$

which we shall prove.

2 Proof:

We start off with three givens:

$$v_0 = v_f \,, \tag{2}$$

and the second is the pair

$$\mathbf{v}_0 \cdot \hat{\mathbf{r}}_0 = -v_0 \qquad \text{and} \qquad \mathbf{v}_f \cdot \hat{\mathbf{r}}_f = v_f \,, \tag{3}$$

and the third is the constant of motion $k\boldsymbol{\epsilon}$ (from Eq. (3.10)):

$$k\boldsymbol{\epsilon} = \mathbf{L}\mathbf{v} - k\hat{\mathbf{r}} \,. \tag{4}$$

Therefore, we know that

$$\mathbf{L}\mathbf{v}_f - k\hat{\mathbf{r}}_f = \mathbf{L}\mathbf{v}_0 - k\hat{\mathbf{r}}_0 \,. \tag{5}$$

From this we get

$$\begin{aligned} (\mathbf{L}v_f - k)\hat{\mathbf{v}}_f &= (-\mathbf{L}v_0 - k)(-\hat{\mathbf{v}}_0) \\ &= (\mathbf{L}v_0 + k)(\hat{\mathbf{v}}_0) \,. \end{aligned} \tag{6}$$

Next, we get

$$(\mathbf{L}v_f - k)\mathbf{v}_f = (\mathbf{L}v_0 + k)\mathbf{v}_0 \,. \tag{7}$$

This gives us

$$\mathbf{v}_f = \left(\frac{\mathbf{L}v_0 + k}{\mathbf{L}v_0 - k} \right) \mathbf{v}_0 \,. \tag{8}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.