Equation (3.24) on Page 211

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1 Introduction

On page 211 of NFCM [1], we find the result (Eq. (3.24))

$$b = \left| \frac{k}{2E} \right| \cot \tfrac{1}{2}\Theta \,, \tag{1}$$

that relates the impact parameter b to the scattering angle Θ.

2 Proof:

We start off with two givens. First, the implicit definition of the scattering angle Θ given by Eq. (3.21):

$$\mathbf{v}_f = \mathbf{v}_0 e^{\mathbf{i}\Theta} \,, \tag{2}$$

and second is Eq. (3.20):

$$\mathbf{v}_f = \left(\frac{\mathbf{L}v_0 + k}{\mathbf{L}v_0 - k} \right) \mathbf{v}_0 \,. \tag{3}$$

Now, the simplest way to make these last two equations compatible is to take the reverse across the last one, using the fact that, given the multivector equation $A = BC$, its reverse is $A^\dagger = (BC)^\dagger = C^\dagger B^\dagger$. So, given that the reverse operation leaves scalars and vectors unchanged then we have that

$$\mathbf{v}_f = \mathbf{v}_0 \left(\frac{\mathbf{L}v_0 + k}{\mathbf{L}v_0 - k} \right)^\dagger = \mathbf{v}_0 \left(\frac{\mathbf{L}^\dagger v_0 + k}{\mathbf{L}^\dagger v_0 - k} \right) = \mathbf{v}_0 \left(\frac{-\mathbf{L}v_0 + k}{-\mathbf{L}v_0 - k} \right) = \mathbf{v}_0 \left(\frac{\mathbf{L}v_0 - k}{\mathbf{L}v_0 + k} \right) . \tag{4}$$

So, comparing this last equation to (2), we get

$$e^{\mathbf{i}\Theta} = \frac{\mathbf{L}v_0 - k}{\mathbf{L}v_0 + k} \,. \tag{5}$$

From here, I'll just use the result given to us in Eq. (3.23):

$$\frac{\mathbf{L}v_0 - k}{\mathbf{L}v_0 + k} = \frac{2Eb\mathbf{i} - |\,k\,|}{2Eb\mathbf{i} + |\,k\,|} \,. \tag{6}$$

Putting these last two equations together, we get

$$(2Eb\mathbf{i} + |\,k\,|)(\cos\Theta + \mathbf{i}\sin\Theta) = 2Eb\mathbf{i} - |\,k\,| \,, \tag{7}$$

which gives us

$$|\,k\,|\cos\Theta - 2Eb\sin\Theta + \mathbf{i}(2Eb\cos\Theta + |\,k\,|\sin\Theta) = 2Eb\mathbf{i} - |\,k\,| \,. \tag{8}$$

The scalar part of this gives us

$$| k | \cos\Theta - 2Eb\sin\Theta = -| k | \,, \tag{9}$$

and the bivector part gives

$$2Eb\cos\Theta + | k | \sin\Theta = 2Eb\,. \tag{10}$$

If now we multiply (9) through by $\cos\Theta$ and multiply (10) through by $\sin\Theta$ and then add together the two resulting equations, we get

$$| k | (1 + \cos\Theta) = 2Eb\sin\Theta\,. \tag{11}$$

which becomes

$$b = \left| \frac{k}{2E} \right| \frac{1 + \cos\Theta}{\sin\Theta} = \left| \frac{k}{2E} \right| \cot\tfrac{1}{2}\Theta\,. \tag{12}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.